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Phases of thermal $\mathcal{N} = 2$ quiver gauge theories

Kasper J. Larsen and Niels A. Obers

The Niels Bohr Institute,
Blegdamsvej 17, 2100 Copenhagen Ø, Denmark
E-mail: kjlarsen@nbi.dk, obers@nbi.dk

Abstract: We consider large N $\mathrm{U}(N)^M$ thermal $\mathcal{N} = 2$ quiver gauge theories on $S^1 \times S^3$. We obtain a phase diagram of the theory with R-symmetry chemical potentials, separating a low-temperature/high-chemical potential region from a high-temperature/low-chemical potential region. In close analogy with the $\mathcal{N} = 4$ SYM case, the free energy is of order $\mathcal{O}(1)$ in the low-temperature region and of order $\mathcal{O}(N^2M)$ in the high-temperature phase. We conclude that the $\mathcal{N} = 2$ theory undergoes a first order Hagedorn phase transition at the curve in the phase diagram separating these two regions. We observe that in the region of zero temperature and critical chemical potential the Hilbert space of gauge invariant operators truncates to smaller subsectors. We compute a 1-loop effective potential with non-zero VEV's for the scalar fields in a sector where the VEV's are homogeneous and mutually commuting. At low temperatures the eigenvalues of these VEV's are distributed uniformly over an $S^5/\mathbb{Z}_M$ which we interpret as the emergence of the $S^5/\mathbb{Z}_M$ factor of the holographically dual geometry $AdS_5 \times S^5/\mathbb{Z}_M$. Above the Hagedorn transition the eigenvalue distribution of the Polyakov loop opens a gap, resulting in the collapse of the joint eigenvalue distribution from $S^5/\mathbb{Z}_M \times S^1$ into $S^6/\mathbb{Z}_M$.

Keywords: 1/N Expansion, AdS-CFT Correspondence, Matrix Models.

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1. Introduction

The phase structure of large N $\mathrm{U}(N)$ gauge theories at finite temperature is in itself a very rich and interesting subject that may provide qualitative insight into the phase structure of QCD. Even more so, the AdS/CFT correspondence [1 − 3] has provided a general framework for translating results obtained in weakly coupled thermal gauge theory into results about the finite temperature behavior of the physics of black holes and stringy geometry at strong coupling. One such connection was suggested by Witten [4] who argued that the Hawking-Page phase transition [5] between thermal AdS_5 and the large AdS_5 Schwarzschild black hole should have a holographic dual description as a confinement/deconfinement transition in the dual thermal field theory defined on the conformal boundary $S^1 \times S^3$ of thermal AdS_5.

A general framework for studying large N $\mathrm{U}(N)$ gauge theories on S^3 at finite temperature was given in [6]. In particular, this considered $\mathcal{N} = 4$ $\mathrm{U}(N)$ SYM theory which was also independently studied in [7]. Furthermore, for the $\mathcal{N} = 4$ case the analysis was extended in [8, 9] to include chemical potentials conjugate to the R-charges. In this way a phase diagram of the theory as a function of both temperature and chemical potentials was obtained. As one application of the phase diagram, in [9] the observation was made that in regions of small temperature and critical chemical potential $\mathcal{N} = 4$ SYM theory reduces to quantum mechanical subsectors, including the $\mathrm{XXX}_{1/2}$ Heisenberg spin chain.[1]

Again for $\mathcal{N} = 4$ SYM theory, the framework of [6, 7] was generalized in a different direction in [11] by allowing non-zero VEV's for the scalar fields. There a one-loop effective potential for the theory at finite temperature on S^3 at weak 't Hooft coupling was computed under the assumption that the VEV's of the scalar fields are constant and diagonal matrices.[2] The potential was used there to study the manifestation of the Gregory-Laflamme instability[3] of the small AdS_5 black hole from the weakly coupled gauge theory point of view. The solutions to the equations of motion obtained from the effective potential of [11] were given in [13] in terms of a joint eigenvalue distribution of the Polyakov loop and the scalar VEV's. Within the sector of constant and commuting scalar VEV's it was found that the topology of the eigenvalue distribution of these VEV's undergoes a phase transition $S^1 \times S^5 \to S^6$ at the Hagedorn temperature. The authors interpreted the S^5 eigenvalue distribution of the scalar VEV's as the emergence of the S^5 factor of the holographically dual thermal $AdS_5 \times S^5$ geometry. It should be noted that, while the truncation to commuting matrices is consistent, this sector will not describe the absolute minima of the action [14]. For this reason the observed phase transitions in the commuting saddles studied in ref. [13] are not transitions in the full gauge theory.

The discovery that the eigenvalues of scalar VEV's reconstruct the dual spacetime geometry was originally made by Berenstein *et al.* in [15 − 18] by setting up matrix models for the various sectors of BPS operators in the chiral ring. In particular the model for 1/8 BPS operators was developed in [17] where the dynamics was shown to reduce to that of

[1] Recently other decoupling limits have been found in near-critical regions by extending this analysis to include the chemical potentials conjugate to the angular momenta on S^3 [10].

[2] This potential was computed earlier in [8] for the special case of zero Polyakov loop eigenvalues.

[3] See [12] for a recent review of the Gregory-Laflamme instability.

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the eigenvalues of three commuting Hermitian matrices Z, X, Y plus two fermionic matrices[4] W_α. The quantum mechanical Hamiltonian for the eigenvalues involves an attractive harmonic oscillator part and a repulsive Vandermonde type part. These forces are balanced when the eigenvalues are localized to a hypersurface in $\mathbb{C}^3$ which is taken to be an S^5 due to the SO(6) invariance of the quantum Hamiltonian. This S^5 was identified with the S^5 factor of the holographically dual geometry $AdS_5 \times S^5$.

The purpose of this paper is to investigate the phase structure of $\mathcal{N} = 2$ $\mathrm{U}(N)^M$ quiver gauge theories at finite temperature.[5] Defined at zero temperature and on a flat spacetime, these gauge theories are $\mathcal{N} = 2$ supersymmetric and conformally invariant [24, 25]. We carry out the investigation of the phase structure in two directions. First, we consider the case of non-zero R-symmetry chemical potentials. One interesting question here is whether the high-temperature phase admits several solutions. A further point of interest is to examine whether one can uncover information about closed subsectors of the as yet not completely settled underlying spin chain of $\mathcal{N} = 2$ quiver gauge theory by studying the near-critical chemical potential and low temperature regions of the (T, μ) phase diagram of the theory as done for $\mathcal{N} = 4$ SYM theory [9, 26].

Another question of interest is to what extent the S^5 eigenvalue distribution of the $\mathcal{N} = 4$ SYM scalar VEV's found in [17] and [13] can be interpreted as the emergence of the S^5 factor of the dual string theory geometry $AdS_5 \times S^5$. To examine this question, we make use of the fact that $\mathcal{N} = 2$ quiver gauge theory can be realized as a $\mathbb{Z}_M$ projection of $\mathcal{N} = 4$ SYM theory. The holographically dual spacetime of the $\mathcal{N} = 2$ theory is thus $AdS_5 \times S^5/\mathbb{Z}_M$ where $\mathbb{Z}_M$ only acts on the S^5 factor. If the above interpretation of emergent spacetime is correct, we should then expect to find an $S^5/\mathbb{Z}_M$ eigenvalue distribution for the VEV's of the scalar fields of $\mathcal{N} = 2$ quiver gauge theory. This has been studied via counting of BPS operators in [27 − 29]. Our approach to the problem is complementary in that it is valid for weak 't Hooft coupling, and it is valid for all temperatures in the range $0 \leq TR \ll \lambda^{-1/2}$ unlike [27 − 29] which is only valid for $T = 0$. In parallel with ref. [13], we restrict to the sector of constant and commuting scalar VEV's. Whereas this enables us to study phase transitions in the eigenvalue distributions, revealing interesting dynamics, it does not necessarily reflect the full phase structure. However, we find it enlightening to see how the geometry of the dual AdS spacetime is mirrored in the structure of the quantum effective action computed in this sector.

The outline and summary of the results in this paper are as follows. In section 2 we give an introduction to $\mathcal{N} = 2$ quiver gauge theory on $S^1 \times S^3$ with chemical potentials conjugate to the R-charges. In section 3 we evaluate the quantum effective action of $\mathcal{N} = 2$ quiver gauge theory with non-zero R-symmetry chemical potentials and zero scalar VEV's in the $g_{\mathrm{YM}} \to 0$ limit and express it in terms of single-particle partition functions. We use the effective action to construct a matrix model for $\mathcal{N} = 2$ quiver gauge theory on $S^1 \times S^3$. The model turns out to be an M-matrix model with adjoint and bifundamental potentials.

In section 4 we study the saddle points of the matrix model as functions of tempera-

[4]However, throughout the analysis of the dynamics in [17], the fermionic matrices W_α are disregarded.

[5]See also refs. [19 − 23] for related work on other supersymmetric gauge theories.

ture and chemical potential and thereby examine the phase structure of the model. In the low-temperature phase we find a saddle point corresponding to a uniform distribution of the eigenvalues of the Polyakov loop.[6] In this phase the free energy is $\mathcal{O}(1)$ with respect to N. This behavior of the free energy suggests that the model in this phase describes a non-interacting gas of color singlet states, and the phase is therefore labelled "confining". This saddle point is observed to become unstable when the temperature is raised above a certain threshold temperature (which depends on the chemical potential). The model then enters a new phase in which the free energy scales as N^2M as $N \to \infty$. This phase is thus interpreted as describing a non-interacting plasma of color non-singlet states and is labelled "deconfined". The "deconfinement" transition is of first order and is identified with a Hagedorn phase transition. The condition of stability of the low-temperature saddle point is translated into a phase diagram of the gauge theory as a function of both temperature and chemical potentials. We subsequently study the phase diagram in regions of small temperature and critical chemical potential. We observe that the Hilbert space of gauge invariant operators truncates to the SU(2) subsector when the chemical potential corresponding to the $\mathrm{SU}(2)_R$ factor of the R-symmetry group $\mathrm{SU}(2)_R \times \mathrm{U}(1)_R$ is turned on, whereas when both chemical potentials are turned on and set equal, it truncates to a larger subsector that corresponds to an orbifolded version of the SU(2|3) sector found in $\mathcal{N} = 4$ SYM theory.

In section 5 we develop a matrix model for $\mathcal{N} = 2$ quiver gauge theory on $S^1 \times S^3$ with non-zero VEV's for the scalar fields and zero R-symmetry chemical potentials. We carry out this computation in the special case where the background fields are assumed to be "commuting" in a sense that conforms to the quiver structure. Furthermore the background fields will be taken to be static and spatially homogeneous in order to preserve the SO(4) isometry of the spatial S^3 manifold. The method employed for computing the effective potential will be the standard background field formalism. That is, we expand the quantum fields about classical background fields and path integrate over the fluctuations, discarding terms of cubic or higher order in the fluctuations. The resulting fluctuation operators turn out to have a particular tridiagonal structure in their quiver indices. By exploiting the vacuum structure of the theory we find that the determinants factorize, leading to an expression for the quantum effective action of $\mathcal{N} = 2$ $\mathrm{U}(N)^M$ quiver gauge theory that explicitly displays the $\mathbb{Z}_M$ structure of the theory. Finally we generalize our results to a specific class of field theories that can be obtained as $\mathbb{Z}_M$ projections of $\mathcal{N} = 4$ SYM theory, of which $\mathcal{N} = 2$ quiver gauge theory is a special case.

In section 6 we find the minima of the matrix model of section 5 in the large N limit in a coarse grained approximation. We consider the joint eigenvalue distribution of the scalar VEV's and the Polyakov loop and find that the topology of the eigenvalue distribution is tied to the Hagedorn phase transition. Below the Hagedorn temperature the eigenvalues of the scalar VEV's are distributed uniformly over an $S^5/\mathbb{Z}_M$ and the eigenvalues of the Polyakov loop are distributed uniformly over an S^1. Thus, the joint eigenvalue distribution

[6]We are using a somewhat sloppy terminology here: by 'Polyakov loop' we really mean the holonomy matrix of a closed curve winding about the thermal circle and not just its trace. Throughout this paper we will use the word to describe both and leave the precise meaning to be determined from the context.

is an $S^5/\mathbb{Z}_M$ fibered trivially over S^1. We interpret this $S^5/\mathbb{Z}_M$ as the emergence of the $S^5/\mathbb{Z}_M$ factor of the holographically dual $AdS_5 \times S^5/\mathbb{Z}_M$ geometry. Above the Hagedorn temperature the eigenvalue distribution of the Polyakov loop becomes gapped and is thus an interval. The scalar VEV's are now distributed uniformly over an $S^5/\mathbb{Z}_M$ fibered over this interval, with the radius of the $S^5/\mathbb{Z}_M$ at any point in the interval proportional to the density of Polyakov loop eigenvalues at that point (for fixed TR). The $S^5/\mathbb{Z}_M$ thus shrinks to zero radius at the endpoints of the interval: the topology of the joint eigenvalue distribution is an $S^6/\mathbb{Z}_M$ where the $\mathbb{Z}_M$ is understood to act on the S^5 transverse to an S^1 diameter. Finally we generalize our results to the $\mathbb{Z}_M$ orbifold field theories discussed at the end of section 5. In particular we find that the geometry of the dual AdS spacetime is mirrored in the structure of the quantum effective action in a precise way within this class of orbifold field theories.

In section 7 we discuss the results we have obtained in this paper and suggest directions for future study. In appendix A further details about $\mathcal{N}=2$ $\mathrm{U}(N)^M$ quiver gauge theory are given, some of which the authors of this paper have not found elsewhere in the literature. In particular, we write the full Lagrangian density in terms of $\mathrm{SU}(2)_R \times \mathrm{U}(1)_R$ invariants. In appendix B we give further technical details of the computation of the quantum effective action obtained in section 5.

2. $\mathcal{N}=2$ quiver gauge theory with R-symmetry chemical potentials

In this section we review $\mathcal{N}=2$ $\mathrm{U}(N)^M$ quiver gauge theories on $S^1 \times S^3$ with R-symmetry chemical potentials. An introductory review of $\mathcal{N}=2$ quiver gauge theories on $S^1 \times S^3$ is given in section 2.1. Details, some of which the authors have not found elsewhere in the literature, are deferred to appendix A. In section 2.2 we then write up the complete Lagrangian density including R-symmetry chemical potentials.

2.1 Review of $\mathcal{N}=2$ quiver gauge theory

$\mathcal{N}=2$ quiver gauge theory with gauge group $\mathrm{U}(N)^M$ arises as the world-volume theory of open strings ending on a stack of N D3-branes placed on the orbifold $\mathbb{C}^3/\mathbb{Z}_M$. The gauge theory is thus superconformal [25] with 16 supercharges. It can be obtained as a $\mathbb{Z}_M$ projection of $\mathcal{N}=4$ $\mathrm{U}(NM)$ SYM theory as explained in detail in appendix A. The resulting gauge group is $\mathrm{U}(N)^M$ where all the $\mathrm{U}(N)$ factors of the gauge group have the same gauge coupling constant g_{YM} associated with them. Letting $i=1,\ldots,M$ and identifying $i \simeq i+M$, the field content can be summarized as follows. There are M vector multiplets[7] $(A_{\mu i}, \Phi_i, \psi_{\Phi,i}, \psi_i)$ where $A_{\mu i}$ is the gauge field, ψ_i is the gaugino, Φ_i is a complex scalar field, and $\psi_{\Phi,i}$ is the superpartner of Φ_i. We take ψ_i and $\psi_{\Phi,i}$ to be 2-component Weyl spinors. Furthermore there are M hypermultiplets $(A_{i,(i+1)}, B_{(i+1),i}, \chi_{A,i}, \chi_{B,i})$ where $A_{i,(i+1)}$ and $B_{(i+1),i}$ are complex scalar fields and $\chi_{A,i}$ and $\chi_{B,i}$ are their respective superpartners which we will take as 2-component Weyl spinors. The fields in the i'th vector multiplet all transform in the adjoint representation of the i'th $\mathrm{U}(N)$ factor of the gauge

[7] We will use an $\mathcal{N}=1$ notation throughout since this proves convenient.

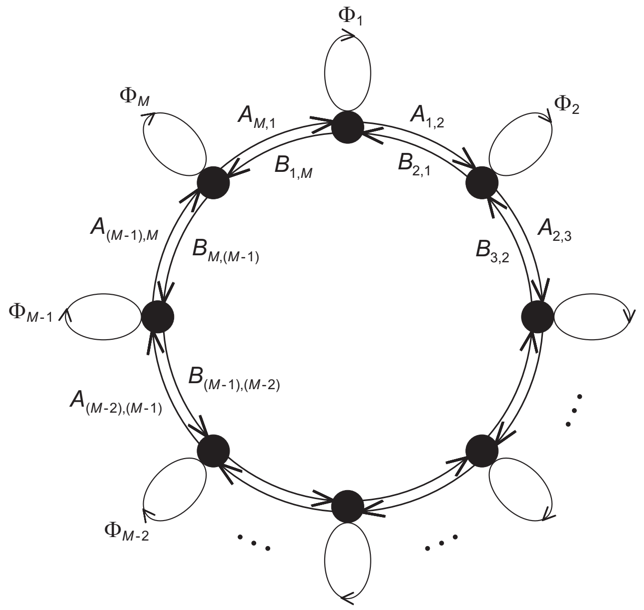

Figure 1: Quiver diagram summarizing the field content of $\mathcal{N}=2$ $\mathrm{U}(N)^M$ quiver gauge theory. Each of the black dots (called nodes) represents a $\mathrm{U}(N)$ gauge group factor. The nodes are labelled by $i=1,\ldots,M$ with the identification $i \simeq i+M$. Arrows go from fundamental to antifundamental representations of the corresponding gauge group factors. The scalar fields $A_{i,(i+1)}$, $B_{(i+1),i}$ and Φ_i are shown in the figure, whereas the gauge fields and all the superpartners have been left implicit.

group. The fields in the i'th hypermultiplet transform in a bifundamental representation of the i'th and $(i+1)$'th factors. More specifically, letting $\mathbf{N}_i$ denote the fundamental representation of the i'th $\mathrm{U}(N)$ factor and $\overline{\mathbf{N}}_i$ the corresponding antifundamental representation, $A_{i,(i+1)}$ and its superpartner $\chi_{A,i}$ transform in the $\mathbf{N}_i \otimes \overline{\mathbf{N}}_{i+1}$ representation, whereas $B_{(i+1),i}$ and its superpartner $\chi_{B,i}$ transform in the $\overline{\mathbf{N}}_i \otimes \mathbf{N}_{i+1}$ representation.

The field content is conveniently summarized in the quiver diagram in figure 1. The diagram consists of M nodes, labelled by $i=1,\ldots,M$ with the identification $i \simeq i+M$. The i'th node represents the i'th $\mathrm{U}(N)$ gauge group factor. Fields belonging to the i'th vector multiplet are drawn as arrows that start and end on the i'th node. For the i'th hypermultiplet, the fields transforming in the $\mathbf{N}_i \otimes \overline{\mathbf{N}}_{i+1}$ representation are drawn as arrows that start at the i'th node and end at the $(i+1)$'th node; the fields transforming in the $\overline{\mathbf{N}}_i \otimes \mathbf{N}_{i+1}$ are depicted as arrows going from the $(i+1)$'th to the i'th node.

The holographic dual of $\mathcal{N}=2$ quiver gauge theory was found in [25] to be Type IIB string theory on $AdS_5 \times S^5/\mathbb{Z}_M$. The quotient $S^5/\mathbb{Z}_M$ is obtained by embedding S^5 in $\mathbb{C}^3$ where the action of $\mathbb{Z}_M$ is as defined in (A.1). The AdS_5 space has a radius given by $R^2_{\mathrm{AdS}} = \sqrt{4\pi g_s (\alpha')^2 NM}$ where g_s is the Type IIB string coupling. There are also NM units of 5-form RR-flux through the AdS_5. Due to the orbifold action the volume of the quotient $S^5/\mathbb{Z}_M$ equals the volume of the covering space S^5 divided by a factor M where the S^5 has the same radius as AdS_5. Similarly, there are N units of 5-form RR-flux through the $S^5/\mathbb{Z}_M$ factor which originate from NM units of flux in the covering space. Finally, we note that the Yang-Mills coupling for each $\mathrm{U}(N)$ gauge group factor g_{YM} is related to the Type IIB coupling by $g_{\mathrm{YM}}^2 = 4\pi g_s M$. This means that the 't Hooft coupling relevant for each factor is $\lambda = g_{\mathrm{YM}}^2 N = 4\pi g_s NM$. This is the same as the 't Hooft coupling on the original NM D3-branes before orbifolding, for which the Yang-Mills coupling was equal to $4\pi g_s$. In the following we will often denote the Yang-Mills coupling simply by g.

The action of $\mathcal{N}=2$ $\mathrm{U}(N)^M$ quiver gauge theory defined on $S^1 \times S^3$ is given as follows. To fix our conventions, we set $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$ and $D_\mu = \partial_\mu + ig[A_\mu, \cdot\,]$. We will denote the circumference of the thermal circle S^1 with β and the radius of the spatial S^3 with R. The Euclidean action of $\mathcal{N}=2$ quiver gauge theory on $S^1 \times S^3$ is then

$$S = \int_{S^1\times S^3} d^4x \sqrt{|g|}\,\left(\mathcal{L}_{\mathrm{gauge}} + \mathcal{L}_{\mathrm{scalar}} + \mathcal{L}_{\mathrm{ferm}}\right) \tag{2.1}$$

where the gauge boson, scalar field and spinor field Lagrangian densities are given by, respectively[8]

$$\begin{aligned}
\mathcal{L}_{\mathrm{gauge}} &= \frac{1}{4}\,\mathrm{Tr}\, F_{\mu\nu}F_{\mu\nu} && (2.2)\\
\mathcal{L}_{\mathrm{scalar}} &= \mathrm{Tr}\,\Big[\Big(D_\mu A\, D_\mu \overline{A} + D_\mu B\, D_\mu \overline{B} + D_\mu \Phi\, D_\mu \overline{\Phi}\Big) \\
&\qquad + R^{-2}\Big(A\overline{A} + B\overline{B} + \Phi\overline{\Phi}\Big) + \frac{1}{2}g^2\Big([A,\overline{A}] + [B,\overline{B}] + [\Phi,\overline{\Phi}]\Big)^2 \\
&\qquad - 2g^2\Big(\big|[A,B]\big|^2 + \big|[A,\Phi]\big|^2 + \big|[B,\Phi]\big|^2\Big)\Big] && (2.3)\\
\mathcal{L}_{\mathrm{ferm}} &= i\,\mathrm{Tr}\,\Big(\overline{\chi_A}\,\tau_\mu \overleftrightarrow{D}_\mu \chi_A + \overline{\chi_B}\,\tau_\mu \overleftrightarrow{D}_\mu \chi_B + \overline{\psi}\,\tau_\mu \overleftrightarrow{D}_\mu \psi + \psi_\Phi\,\tau_\mu \overleftrightarrow{D}_\mu \overline{\psi_\Phi}\Big) \\
&\quad + \frac{g}{\sqrt{2}}\,\mathrm{Tr}\,\Big(\overline{\chi_A}\big([A,\psi_\Phi] - [\overline{B},\overline{\psi}]\big) + \overline{\chi_B}\big([\overline{A},\overline{\psi}] + [B,\psi_\Phi]\big) \\
&\qquad\qquad - \overline{\psi}\big([\overline{A},\overline{\chi_B}] - [\overline{B},\overline{\chi_A}]\big) - \psi_\Phi\big([A,\overline{\chi_A}] + [B,\overline{\chi_B}]\big) \\
&\qquad\qquad + \chi_A\big([\overline{A},\overline{\psi_\Phi}] - [B,\psi]\big) + \chi_B\big([A,\psi] + [\overline{B},\overline{\psi_\Phi}]\big) \\
&\qquad\qquad - \psi\big([A,\chi_B] - [B,\chi_A]\big) - \overline{\psi_\Phi}\big([\overline{A},\chi_A] + [\overline{B},\chi_B]\big) \\
&\qquad\qquad + \overline{\chi_A}\,[\overline{\Phi},\overline{\chi_B}] - \overline{\chi_B}\,[\overline{\Phi},\overline{\chi_A}] + \overline{\psi}\,[\Phi,\psi_\Phi] - \psi_\Phi\,[\Phi,\overline{\psi}] \\
&\qquad\qquad + \chi_A\,[\Phi,\chi_B] - \chi_B\,[\Phi,\chi_A] + \psi\,[\overline{\Phi},\overline{\psi_\Phi}] - \overline{\psi_\Phi}\,[\overline{\Phi},\psi]\Big)\,. && (2.4)
\end{aligned}$$

The traces are taken over the $NM\times NM$ matrices. The spinor fields $\chi_A, \chi_B, \psi_\Phi, \psi$ are undotted 2-component Weyl spinors. We define $\tau_\mu = (1, i\boldsymbol{\sigma})$. The operator $\overleftrightarrow{D}_\mu$ is defined by $\psi_1 \overleftrightarrow{D}_\mu \psi_2 \equiv \frac{1}{2}\big(\psi_1 D_\mu \psi_2 - (D_\mu\psi_1)\,\psi_2\big)$. It is implied that the fields A, B, Φ, A_μ etc. take the orbifold projection invariant forms given in eqs. (A.15)–(A.16) and (A.31)–(A.32). Note that the scalar fields are conformally coupled to the curvature of the spatial manifold S^3 through the term $R^{-2}\,\mathrm{Tr}\,\big(A\overline{A} + B\overline{B} + \Phi\overline{\Phi}\big)$ in (2.3). This effectively induces a mass for the scalar fields.

[8]Note that for all fields, including the Weyl spinors $\chi_A, \chi_B, \psi_\Phi, \psi$, the bars denote the *Hermitian* conjugate, not the complex or Weyl conjugate. E.g., $(\overline{\chi_A})_{\alpha\beta} = (\chi_A)^*_{\beta\alpha}$ where α,β are gauge group indices and the * denotes complex conjugation. Furthermore, in the third line of eq. (2.3), the notation means, e.g., $|[A,B]|^2 \equiv [A,B][\overline{A},\overline{B}]$.

The orbifolding breaks the R-symmetry group SU(4) of $\mathcal{N} = 4$ SYM theory into $\mathrm{SU}(2)_R \times \mathrm{U}(1)_R$. As described in appendix A, Φ is associated with the z_1 direction of $\mathbb{C}^3$ which is inert under the action of the orbifold group $\mathbb{Z}_M$, while A and B are associated with z_2 and z_3 respectively. The $\mathrm{U}(1)_R$ factor corresponds to the transformation $z_1 \to e^{i\zeta} z_1$ and therefore acts on the Φ fields by multiplying phase rotations. The A and B fields have zero charge under $\mathrm{U}(1)_R$. The $\mathrm{SU}(2)_R$ symmetry acts on the A and B fields and their Hermitian conjugates. In fact, $(A, \overline{B})$ and $(-B, \overline{A})$ form $\mathrm{SU}(2)_R$ doublets. Furthermore $(\overline{\psi}, \psi_\Phi)$ and $(-\overline{\psi_\Phi}, \psi)$ are $\mathrm{SU}(2)_R$ doublets whereas χ_A and χ_B have zero charge under $\mathrm{SU}(2)_R$. The gauge field is not charged under $\mathrm{SU}(2)_R \times \mathrm{U}(1)_R$. We summarize the R-charges in table 2.

2.2 Lagrangian density with R-symmetry chemical potentials

Given any non-Abelian symmetry group G, one can introduce chemical potentials conjugate to the generators of a maximal torus of G. In this section we will consider the case where G is the R-symmetry group $\mathrm{SU}(2)_R \times \mathrm{U}(1)_R$ of $\mathcal{N} = 2$ quiver gauge theory. The maximal torus is $\mathrm{U}(1) \times \mathrm{U}(1)$. We will denote the Cartan generators of $\mathrm{U}(1)_R$ and $\mathrm{SU}(2)_R$ by Q_1 and Q_2, respectively, and the corresponding chemical potentials by μ_1 and μ_2. For the $\mathrm{U}(1)$ factor of the maximal torus that corresponds to $\mathrm{U}(1)_R$ the eigenvalues of the Cartan generators can directly be read off from table 2. For the $\mathrm{U}(1) \subset \mathrm{SU}(2)_R$ we choose as a basis for the Cartan subalgebra the diagonal generator σ_z so that the $\mathrm{SU}(2)_R$ doublets will have well-defined charges under $\mathrm{U}(1)$. (We choose σ_z rather than $\frac{1}{2}\sigma_z$ as the generator Q_2 because we require $e^{iQ_2\theta}$ to be invariant under $\theta \to \theta + 2\pi$. Setting $Q_2 \equiv \sigma_z$ we have $e^{iQ_2\theta} = \mathrm{diag}(e^{i\theta}, e^{-i\theta})$ which is clearly invariant.) Therefore the charges under the maximal torus $\mathrm{U}(1)$ of $\mathrm{SU}(2)_R$ will be 2 times the $\mathrm{SU}(2)_R$ charges.

Thus for the bosonic fields,

$$(\mu_a Q_a)\, A_{i,(i+1)} = \mu_2\, A_{i,(i+1)} \tag{2.5}$$

$$(\mu_a Q_a)\, B_{(i+1),i} = \mu_2\, B_{(i+1),i} \tag{2.6}$$

$$(\mu_a Q_a)\, \Phi_i = \mu_1 \Phi_i \tag{2.7}$$

$$(\mu_a Q_a)\, A_{\mu i} = 0\,, \tag{2.8}$$

and for the fermionic fields,

$$(\mu_a Q_a)\, \chi_{A,i} = -\frac{1}{2}\mu_1\, \chi_{A,i} \tag{2.9}$$

$$(\mu_a Q_a)\, \chi_{B,i} = -\frac{1}{2}\mu_1\, \chi_{B,i} \tag{2.10}$$

$$(\mu_a Q_a)\, \psi_i = \left(\frac{1}{2}\mu_1 - \mu_2\right) \psi_i \tag{2.11}$$

$$(\mu_a Q_a)\, \psi_{\Phi,i} = \left(-\frac{1}{2}\mu_1 - \mu_2\right) \psi_{\Phi,i}\,. \tag{2.12}$$

The corresponding expressions for the Hermitian conjugate fields are obtained by simply changing the signs of the chemical potentials.

To obtain the Lagrangian density of $\mathcal{N} = 2$ quiver gauge theory with chemical potentials μ_a for the $\mathrm{SU}(2)_R \times \mathrm{U}(1)_R$ Cartan generators, one makes the following substitution

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in the Lagrangian density

$$D_\mu \longrightarrow D_\mu - \mu_a Q_a \delta_{\mu 0}\ . \tag{2.13}$$

Below we have written the Lagrangian densities for the fundamental scalar and spinor fields of $\mathcal{N}=2$ quiver gauge theory. This will be important for the analysis in the following sections in order to distinguish the adjoint from the bifundamental structures.

The Lagrangian density for the scalar fields with R-symmetry chemical potentials is

$$\begin{aligned}
\mathcal{L}_{\rm scalar} = \sum_{i=1}^{M} \Bigg\{ & \operatorname{Tr}\Big[\Big(\partial_\mu A_{i,(i+1)} + igA_{\mu i}A_{i,(i+1)} - igA_{i,(i+1)}A_{\mu(i+1)} - \mu_2\delta_{\mu 0}A_{i,(i+1)}\Big) \\
& \qquad \times \Big(\partial_\mu \overline{A_{i,(i+1)}} + igA_{\mu(i+1)}\overline{A_{i,(i+1)}} - ig\overline{A_{i,(i+1)}}A_{\mu i} + \mu_2\delta_{\mu 0}\overline{A_{i,(i+1)}}\Big)\Big] \\
& + \operatorname{Tr}\Big[\Big(\partial_\mu B_{(i+1),i} + igA_{\mu(i+1)}B_{(i+1),i} - igB_{(i+1),i}A_{\mu i} - \mu_2\delta_{\mu 0}B_{(i+1),i}\Big) \\
& \qquad \times \Big(\partial_\mu \overline{B_{(i+1),i}} + igA_{\mu i}\overline{B_{(i+1),i}} - ig\overline{B_{(i+1),i}}A_{\mu(i+1)} + \mu_2\delta_{\mu 0}\overline{B_{(i+1),i}}\Big)\Big] \\
& + \operatorname{Tr}\Big[\Big(\partial_\mu\Phi_i + ig[A_{\mu i},\Phi_i] - \mu_1\delta_{\mu 0}\Phi_i\Big)\Big(\partial_\mu\overline{\Phi_i} + ig[A_{\mu i},\overline{\Phi_i}] + \mu_1\delta_{\mu 0}\overline{\Phi_i}\Big)\Big] \\
& + R^{-2}\operatorname{Tr}\Big(A_{i,(i+1)}\overline{A_{i,(i+1)}} + \overline{B_{(i+1),i}}B_{(i+1),i} + \Phi_i\overline{\Phi_i}\Big) \\
& + \frac{1}{2}g^2\operatorname{Tr}\Big[\Big(A_{i,(i+1)}\overline{A_{i,(i+1)}} - \overline{A_{(i-1),i}}A_{(i-1),i} \\
& \qquad\qquad + B_{i,(i-1)}\overline{B_{i,(i-1)}} - \overline{B_{(i+1),i}}B_{(i+1),i} + [\Phi_i,\overline{\Phi_i}]\Big)^2\Big] \\
& - 2g^2\operatorname{Tr}\Big[\Big(A_{i,(i+1)}B_{(i+1),i} - B_{i,(i-1)}A_{(i-1),i}\Big) \\
& \qquad \times \Big(\overline{A_{(i-1),i}}\ \overline{B_{i,(i-1)}} - \overline{B_{(i+1),i}}\ \overline{A_{i,(i+1)}}\Big)\Big] \\
& - 2g^2\operatorname{Tr}\Big[\Big(A_{i,(i+1)}\Phi_{i+1} - \Phi_i A_{i,(i+1)}\Big)\Big(\overline{A_{i,(i+1)}}\ \overline{\Phi_i} - \overline{\Phi_{i+1}}\ \overline{A_{i,(i+1)}}\Big)\Big] \\
& - 2g^2\operatorname{Tr}\Big[\Big(B_{(i+1),i}\Phi_i - \Phi_{i+1}B_{(i+1),i}\Big)\Big(\overline{B_{(i+1),i}}\ \overline{\Phi_{i+1}} - \overline{\Phi_i}\ \overline{B_{(i+1),i}}\Big)\Big]\Bigg\}.
\end{aligned} \tag{2.14}$$

Here the traces are always taken over the gauge indices of the $N\times N$ matrices. Observe that the chemical potentials μ_1 and μ_2 act like negative mass squares for Φ_i and $A_{i,(i+1)}, B_{(i+1),i}$. On a compact spatial manifold such as S^3, these terms are balanced by the positive mass square terms induced by the conformal coupling to curvature. We immediately observe from (2.14) that $\mathcal{N}=2$ quiver gauge theory on $S^1\times S^3$ is well-defined as long as $\mu_1,\mu_2 \le R^{-1}$. If the chemical potentials exceed this bound, the theory develops tachyonic modes and there exists no stable ground state.

The Lagrangian density for the spinor fields with R-symmetry chemical potentials is

$$\begin{aligned}
\mathcal{L}_{\rm ferm} = \sum_{i=1}^{M} \Bigg\{ & \frac{i}{2}\operatorname{Tr}\Big(\overline{\chi_{A,i}}\,\tau_\mu\big(\partial_\mu\chi_{A,i} + igA_{\mu i}\chi_{A,i} - ig\chi_{A,i}A_{\mu(i+1)} + \frac{1}{2}\mu_1\delta_{\mu 0}\chi_{A,i}\big)\Big) \\
& -\frac{i}{2}\operatorname{Tr}\Big(\big(\partial_\mu\overline{\chi_{A,i}} + igA_{\mu(i+1)}\overline{\chi_{A,i}} - ig\overline{\chi_{A,i}}A_{\mu i} - \frac{1}{2}\mu_1\delta_{\mu 0}\overline{\chi_{A,i}}\big)\,\tau_\mu\,\chi_{A,i}\Big) \\
& +\frac{i}{2}\operatorname{Tr}\Big(\overline{\chi_{B,i}}\,\tau_\mu\big(\partial_\mu\chi_{B,i} + igA_{\mu(i+1)}\chi_{B,i} - ig\chi_{B,i}A_{\mu i} + \frac{1}{2}\mu_1\delta_{\mu 0}\chi_{B,i}\big)\Big)
\end{aligned}$$

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$$
\begin{aligned}
&-\frac{i}{2}\,\mathrm{Tr}\left(\left(\partial_\mu\overline{\chi_{B,i}}+igA_{\mu i}\overline{\chi_{B,i}}-ig\overline{\chi_{B,i}}A_{\mu(i+1)}-\frac{1}{2}\mu_1\delta_{\mu 0}\overline{\chi_{B,i}}\right)\tau_\mu\,\chi_{B,i}\right)\\
&+\frac{i}{2}\,\mathrm{Tr}\left(\overline{\psi_i}\,\tau_\mu\left(\partial_\mu\psi_i+ig[A_{\mu i},\psi_i]-\left(\frac{1}{2}\mu_1-\mu_2\right)\delta_{\mu 0}\psi_i\right)\right)\\
&-\frac{i}{2}\,\mathrm{Tr}\left(\left(\partial_\mu\overline{\psi_i}+ig[A_{\mu i},\overline{\psi_i}]+\left(\frac{1}{2}\mu_1-\mu_2\right)\delta_{\mu 0}\overline{\psi_i}\right)\tau_\mu\,\psi_i\right)\\
&+\frac{i}{2}\,\mathrm{Tr}\left(\psi_{\Phi,i}\,\tau_\mu\left(\partial_\mu\overline{\psi_{\Phi,i}}+ig[A_{\mu i},\overline{\psi_{\Phi,i}}]-\left(\frac{1}{2}\mu_1+\mu_2\right)\delta_{\mu 0}\overline{\psi_{\Phi,i}}\right)\right)\\
&-\frac{i}{2}\,\mathrm{Tr}\left(\left(\partial_\mu\psi_{\Phi,i}+ig[A_{\mu i},\psi_{\Phi,i}]+\left(\frac{1}{2}\mu_1+\mu_2\right)\delta_{\mu 0}\psi_{\Phi,i}\right)\tau_\mu\,\overline{\psi_{\Phi,i}}\right)\\
&+\frac{g}{\sqrt{2}}\,\mathrm{Tr}\left(\epsilon^{cd}\left\{\chi_{A,i}(\overline{\lambda_i})_c,\,(\overline{\chi_i})_d\right\}+\epsilon^{cd}\left\{\chi_{A,i},\,(\overline{\chi_{i+1}})_c(\overline{\lambda_i})_d\right\}\right.\\
&\qquad+\epsilon^{cd}\left\{\overline{\chi_{A,i}}(\lambda_i)_c,\,(\chi_{i+1})_d\right\}+\epsilon^{cd}\left\{\overline{\chi_{A,i}},\,(\chi_i)_c(\lambda_i)_d\right\}\\
&\qquad+\epsilon^{cd}\left\{\chi_{B,i}(\lambda_i)_c,\,(\overline{\chi_{i+1}})_d\right\}+\epsilon^{cd}\left\{\chi_{B,i},\,(\overline{\chi_i})_c(\lambda_i)_d\right\}\\
&\qquad-\epsilon^{cd}\left\{\overline{\chi_{B,i}}(\overline{\lambda_i})_c,\,(\chi_i)_d\right\}-\epsilon^{cd}\left\{\overline{\chi_{B,i}},\,(\chi_{i+1})_c(\overline{\lambda_i})_d\right\}\\
&\qquad+\epsilon^{cd}\left\{(\chi_i)_c\Phi_i,\,(\chi_i)_d\right\}+\epsilon^{cd}\left\{(\overline{\chi_i})_c\overline{\Phi_i},\,(\overline{\chi_i})_d\right\}\\
&\qquad+\left\{\chi_{A,i}\Phi_{i+1},\,\chi_{B,i}\right\}+\left\{\overline{\chi_{A,i}}\,\overline{\Phi_i},\,\overline{\chi_{B,i}}\right\}\\
&\qquad\left.-\left\{\chi_{B,i}\Phi_i,\,\chi_{A,i}\right\}-\left\{\overline{\chi_{B,i}}\,\overline{\Phi_{i+1}},\,\overline{\chi_{A,i}}\right\}\right)\Bigg\}\,.
\end{aligned}
\tag{2.15}
$$

Here the traces are always taken over the gauge indices of the $N \times N$ matrices. Note that the potential part of the Lagrangian density has been written in terms of the $\mathrm{SU}(2)_R$ doublets given in eqs. (A.43)–(A.44) for notational simplicity.

Finally, as the gauge fields have zero charge under $\mathrm{SU}(2)_R\times\mathrm{U}(1)_R$, the gauge field part of the Lagrangian density is unaffected by introducing the R-symmetry chemical potentials. Nonetheless, we give the result here for convenience:

$$
\mathcal{L}_{\text{gauge}} = \frac{1}{4}\sum_{i=1}^{M}\mathrm{Tr}\,F^i_{\mu\nu}F^i_{\mu\nu} \tag{2.16}
$$

where of course $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + ig[A^i_\mu, A^i_\nu]$ and the trace is taken over the gauge indices of the $N \times N$ matrices.

3. Zero-coupling limit and the matrix model

The matrix model we will consider is defined by integrating out the fluctuations of the quantum fields. In section 3.1 we therefore first give a brief description of how to compute the one-loop quantum effective action with non-zero chemical potentials conjugate to the R-charges. The details of this computation are well-described in the literature (see, e.g., appendix A of [8]). In section 3.2 we then proceed to construct the matrix model out of the 1-loop quantum effective action.

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3.1 One-loop quantum effective action

The partition function for the grand canonical ensemble has the path integral representation

$$Z = \int \mathcal{D}A_\mu \, \mathcal{D}\phi \, \mathcal{D}\psi \, e^{-\int_{S^1\times S^3} d^4x \, \sqrt{|g|} \, (\mathcal{L}_{\text{gauge}}+\mathcal{L}_{\text{scalar}}+\mathcal{L}_{\text{ferm}})} \tag{3.1}$$

with $\mathcal{L}_{\text{gauge}}, \mathcal{L}_{\text{scalar}}$ and $\mathcal{L}_{\text{ferm}}$ being the Lagrangian densities with R-symmetry chemical potentials given by eqs. (2.16), (2.14) and (2.15), respectively, and where the measures $\mathcal{D}A_\mu$, $\mathcal{D}\phi$ and $\mathcal{D}\psi$ are the products of the measures over all the gauge fields, scalar fields and spinor fields, respectively. We will obtain an effective action from this expression by taking the free limit $g \to 0$ of the tree-level action. However, since the theory is defined on a compact spatial S^3 one must impose the Gauss law constraint that all states be gauge invariant. We perform the projection onto gauge invariant states by using A_{0i} as a Lagrange multiplier,

$$A_{\mu i}(x) \longrightarrow \widetilde{A}_{\mu i}(x) + \delta_{\mu 0} a_i/g \tag{3.2}$$

where $\widetilde{A}_{0i}$ integrates to zero over $S^1 \times S^3$ and a_i are constant Hermitian matrices which by gauge invariance can be assumed diagonal, $a_i = \text{diag}(q_i^1, \ldots, q_i^N)$. To obtain the correct zero coupling limit one inserts the decomposition (3.2) into the action given through (2.14)–(2.16) and then takes the $g \to 0$ limit.

As the quantum fields are defined on $S^1 \times S^3$ one decomposes them into Fourier modes on S^1 and S^3 spherical harmonics. More specifically, let τ denote the direction along the S^1. We will use the convention that any field ϕ defined on $S^1 \times S^3$ has the Fourier mode decomposition

$$\phi(\tau, \boldsymbol{x}) = \sum_{k=-\infty}^{\infty} e^{i\omega_k \tau} \phi^{[k]}(\boldsymbol{x}) \tag{3.3}$$

where the quantized Matsubara frequencies are $\omega_k = \frac{2\pi k}{\beta}$ for bosons and $\omega_k = \frac{(2k+1)\pi}{\beta}$ for fermions giving, respectively, periodic and antiperiodic boundary conditions around the thermal circle.[9] One then decomposes the spatial components of the gauge field into spherical harmonics on S^3 by writing them as a sum of a transverse (i.e. divergenceless) vector field $\mathbf{A}_i^\perp$ and a longitudinal vector field ∇F_i where F_i is a scalar function. That is, for $k = 1, 2, 3$ we decompose

$$\widetilde{A}_i^k = (A_i^\perp)^k + (\nabla F_i)^k \tag{3.4}$$

and insert the expression on the right hand side into the action given through (2.14)–(2.16).

The quantum effective action $\Gamma \equiv -\ln Z$ is defined by integrating out all fluctuating fields (cf. (3.1)), leaving an expression that only depends on the zero mode a_i. It is convenient to express Γ as a functional of the holonomy matrix of a closed curve wound around the thermal circle, i.e. $U_i \equiv e^{i\beta a_i}$ after decomposing the gauge field according to (3.2) and taking $g \to 0$. By performing the traces over the Matsubara frequencies and over the angular momenta h, with appropriate eigenvalues of the Laplacian ∇^2 on S^3 and

[9] However, for the Fadeev-Popov ghosts the boundary conditions are taken periodic.

quantum field		eigenvalue	notation in text	degeneracy (D_h)
transverse vector	$\mathbf{A}^{\perp}$	$-(h+1)^2\,R^{-2}$	$-\Delta_g^2$	$2h(h+2)$
longitudinal vector	∇F	$-h(h+2)\,R^{-2}$	$-\Delta_s^2$	$(h+1)^2$
real scalar	A^0, ϕ	$-h(h+2)\,R^{-2}$	$-\Delta_s^2$	$(h+1)^2$
Weyl spinor	ψ	$-\left(h+\frac{1}{2}\right)^2 R^{-2}$	$-\Delta_f^2$	$h(h+1)$

Table 1: Eigenvalues and corresponding degeneracies of the S^3 spatial Laplacian $\nabla^2 \equiv \partial_1^2+\partial_2^2+\partial_3^2$ for various quantum fields defined on S^3. Here R denotes the radius of S^3. The irreducible representations of the SO(4) isometry group are labelled by the angular momentum h which has the range $h = 0, 1, 2, \ldots$ for all the fields except for the longitudinal vector field ∇F where h starts from 1.

the associated degeneracies (cf. table 1) one finds the following expression for the quantum effective action in terms of the variables $x \equiv e^{-\beta}$ and $y_j \equiv e^{\beta\mu_j}$

$$\begin{aligned}
\Gamma[U_i] = &-\sum_{i=1}^{M}\sum_{l=1}^{\infty}\left[\frac{1}{l}\left(\frac{6x^{2l}-2x^{3l}}{(1-x^l)^3}\right)+\frac{1}{l}\left(\frac{x^l+x^{2l}}{(1-x^l)^3}\right)\left(y_1^l+y_1^{-l}\right)\right.\\
&\qquad\left.+\frac{(-1)^{l+1}}{l}\left(\frac{2x^{3l/2}}{(1-x^l)^3}\right)\left(y_1^{l/2}+y_1^{-l/2}\right)\left(y_2^l+y_2^{-l}\right)\right]\left(\operatorname{Tr}U_i^l\,\operatorname{Tr}U_i^{-l}\right)\\
&-\sum_{i=1}^{M}\sum_{l=1}^{\infty}\left[\frac{1}{l}\left(\frac{x^l+x^{2l}}{(1-x^l)^3}\right)\left(y_2^l+y_2^{-l}\right)+\frac{(-1)^{l+1}}{l}\left(\frac{2x^{3l/2}}{(1-x^l)^3}\right)\left(y_1^{l/2}+y_1^{-l/2}\right)\right]\\
&\qquad\times\left(\operatorname{Tr}U_i^l\,\operatorname{Tr}U_{i+1}^{-l}+\operatorname{Tr}U_i^{-l}\,\operatorname{Tr}U_{i+1}^{l}\right).
\end{aligned}\tag{3.5}$$

Note that the adjoint holonomy factors come from the vector multiplets $\left(A_{\mu i}, \Phi_i, \psi_{\Phi,i}, \psi_i\right)$, and the bifundamental factors come from the hypermultiplets $\left(A_{i,(i+1)}, B_{(i+1),i}, \chi_{A,i}, \chi_{B,i}\right)$. For later convenience we define here the total single-particle partition functions for the bosonic and fermionic sectors of the vector and hypermultiplets:

$$z_{\rm ad}^B(x;y_1,y_2) \equiv \frac{6x^2-2x^3}{(1-x)^3}+\frac{x+x^2}{(1-x)^3}\left(y_1+y_1^{-1}\right)\tag{3.6}$$

$$z_{\rm ad}^F(x;y_1,y_2) \equiv \frac{2x^{3/2}}{(1-x)^3}\left(y_1^{1/2}+y_1^{-1/2}\right)\left(y_2+y_2^{-1}\right)\tag{3.7}$$

$$z_{\rm bi}^B(x;y_1,y_2) \equiv \frac{x+x^2}{(1-x)^3}\left(y_2+y_2^{-1}\right)\tag{3.8}$$

$$z_{\rm bi}^F(x;y_1,y_2) \equiv \frac{2x^{3/2}}{(1-x)^3}\left(y_1^{1/2}+y_1^{-1/2}\right).\tag{3.9}$$

These results are consistent with ref. [6], eqs. (3.17)-(3.18), where the summation over representations is taken to run over the adjoint and the bifundamental representations, and the charges Q are taken as β times the Cartan charges Q_1, Q_2 given implicitly through (2.5)–(2.12).

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3.2 The matrix model

The matrix model we will consider is defined by the partition function

$$Z_{\rm MM} = \int \prod_{i=1}^{M} [\mathcal{D}U_i] \exp\big(-\Gamma[U_i]\big) \tag{3.10}$$

where $\Gamma[U_i]$ is given in (3.5). It is convenient for taking the continuum limit to rewrite $\Gamma[U_i]$ directly in terms of the zero modes a_i. To simplify the notation, define the rescaled zero mode $\alpha_i \equiv \beta a_i$ so that $U_i = e^{i\alpha_i}$. Hence

$$Z_{\rm MM} = \int \prod_{i=1}^{M} [\mathcal{D}\alpha_i] \; \exp\left(- \sum_{m\neq n} \Big(V_{\rm ad}(\alpha_i^m - \alpha_i^n) + V_{\rm bi}(\alpha_i^m - \alpha_{i+1}^n)\Big)\right) \tag{3.11}$$

where the adjoint and bifundamental potentials are, respectively

$$\begin{aligned} V_{\rm ad}(\theta) &\equiv -\ln\left|\sin\left(\frac{\theta}{2}\right)\right| - \sum_{l=1}^{\infty}\frac{1}{l}\,\Big(z_{\rm ad}^B(x^l;y_1^l,y_2^l) + (-1)^{l+1} z_{\rm ad}^F(x^l;y_1^l,y_2^l)\Big)\cos(l\theta) \\ &= \ln 2 + \sum_{l=1}^{\infty}\frac{1}{l}\Big(1 - z_{\rm ad}^B(x^l;y_1^l,y_2^l) - (-1)^{l+1} z_{\rm ad}^F(x^l;y_1^l,y_2^l)\Big)\cos(l\theta) \end{aligned} \tag{3.12}$$

$$V_{\rm bi}(\theta) \equiv -\sum_{l=1}^{\infty}\frac{2}{l}\,\Big(z_{\rm bi}^B(x^l;y_1^l,y_2^l) + (-1)^{l+1} z_{\rm bi}^F(x^l;y_1^l,y_2^l)\Big)\cos(l\theta)\,. \tag{3.13}$$

We will now take the continuum limit $N\to\infty$. It is convenient to introduce eigenvalue distributions $\rho_i(\theta_i)$ proportional to the density of the eigenvalues $e^{i\theta_i}$ of U_i at the angle $\theta_i \in [-\pi,\pi]$. Here ρ_i must be everywhere non-negative, and we choose its normalization so that for any fixed i

$$\int_{-\pi}^{\pi} d\theta_i\, \rho_i(\theta_i) = 1\,. \tag{3.14}$$

Furthermore we define the Fourier modes of ρ_i and $V_{\rm ad}$ and $V_{\rm bi}$:

$$\rho_i^l \equiv \int_{-\pi}^{\pi} d\theta_i\, \rho_i(\theta_i)\cos(l\theta_i)\,, \quad V_{\rm ad}^l \equiv \int_{-\pi}^{\pi} d\theta\, V_{\rm ad}(\theta)\cos(l\theta)\,, \quad V_{\rm bi}^l \equiv \int_{-\pi}^{\pi} d\theta\, V_{\rm bi}(\theta)\cos(l\theta) \tag{3.15}$$

so that, assuming ρ_i, $V_{\rm ad}$, $V_{\rm bi}$ to be even functions, we have the Fourier expansions

$$\rho_i(\zeta) = \frac{1}{\pi}\sum_{l=0}^{\infty}\rho_i^l\cos(l\zeta), \quad V_{\rm ad}(\zeta) = \frac{1}{\pi}\sum_{l=0}^{\infty}V_{\rm ad}^l\cos(l\zeta), \quad V_{\rm bi}(\zeta) = \frac{1}{\pi}\sum_{l=0}^{\infty}V_{\rm bi}^l\cos(l\zeta). \tag{3.16}$$

The continuum limit is obtained by making the substitution[10]

$$\frac{1}{N}\sum_{n=1}^{N}[\cdots] \;\longrightarrow\; \int_{-\pi}^{\pi} d\theta_i\,\rho_i(\theta_i)\,[\cdots] \tag{3.17}$$

[10]Here it is implied that the content of the brackets $[\cdots]$ carries an i label.

Furthermore we replace the path integral measure $\left[\mathcal{D}\alpha_i\right] \longrightarrow \left[\mathcal{D}\lambda_i\right]$. Thus, in the continuum limit the path integral of the matrix model takes the form

$$Z_{\mathrm{MM}} = \int \prod_{i=1}^{M} \left[\mathcal{D}\lambda_i\right] \exp\left(-S_{\mathrm{MM}}[\boldsymbol{\rho}]\right) \tag{3.18}$$

where the action for the eigenvalue distribution functions $\boldsymbol{\rho}$ is

$$S_{\mathrm{MM}}[\boldsymbol{\rho}] = \frac{N^2}{\pi} \sum_{i=1}^{M} \sum_{l=1}^{\infty} \Big((\rho_i^l)^2 V_{\mathrm{ad}}^l(T;\mu_1,\mu_2) \; + \; \rho_i^l \rho_{i+1}^l V_{\mathrm{bi}}^l(T;\mu_1,\mu_2) \Big)\,. \tag{3.19}$$

To summarize, the matrix model under study is defined by eqs. (3.18)–(3.19).

4. Phase structure

The term $-\ln\left|\sin\left(\frac{\theta}{2}\right)\right|$ in the adjoint potential (3.12) originating from the change of measure is a temperature-independent repulsive potential. On the other hand, the remaining parts of the adjoint and bifundamental potentials (3.12)–(3.13) provide an attractive force[11] which grows from zero to infinite strength as the temperature is raised from zero to infinity. One would therefore expect that at low temperatures, the stable saddle points of the matrix model are characterized by the eigenvalues of the holonomy matrices U_i spreading out uniformly over the unit circle, whereas at high temperatures the attractive potential causes them to localize [6].

4.1 Low-temperature solution and phase transition

We now consider the saddle points of the matrix model action (3.19),

$$0 = \frac{\partial S_{\mathrm{MM}}}{\partial \rho_i^l} = \frac{N^2}{\pi}\Big(2\rho_i^l V_{\mathrm{ad}}^l \; + \; \big(\rho_{i-1}^l + \rho_{i+1}^l\big) V_{\mathrm{bi}}^l\Big)\,. \tag{4.1}$$

For $M \geq 2$, this condition translates into M linear equations in M unknowns:

$$2\rho_i^l V_{\mathrm{ad}}^l \; + \; \big(\rho_{i-1}^l + \rho_{i+1}^l\big) V_{\mathrm{bi}}^l = 0\,. \tag{4.2}$$

The determinant of this system of equations is generically non-zero, so we find the unique solution $\rho_i^l = 0$, corresponding to the flat distribution $\rho_i = \frac{1}{2\pi}$. Thus we conclude that the eigenvalues of the holonomy matrices U_i are distributed uniformly on each of the M unit circles. This defines the low-temperature solution of the matrix model.

The leading $\mathcal{O}(N^2)$ contribution to the free energy computed from the path integral (3.18) comes from the action $S_{\mathrm{MM}}[\boldsymbol{\rho}]$. However, as $\rho_i^l = 0$, the first non-zero contribution to the free energy in this phase comes from a Gaussian integral over the fluctuations about the solution $\rho_i = \frac{1}{2\pi}$. The free energy is therefore of $\mathcal{O}(1)$ with respect to N, suggesting that the theory in this phase describes a non-interacting gas of color singlet states.

[11] The fact that the remaining parts of (3.12)–(3.13) are attractive potentials can be shown following the argument in [6], footnote 32.

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Furthermore, we note that the Polyakov loop $W(C) \equiv \mathrm{Tr}\,\mathcal{P}\exp\left(ig\int_0^\beta dx^0 A_i^0\right)$ has zero expectation value since the trace averages to zero in the uniform eigenvalue distribution. In particular, this implies that the $\mathbb{Z}_N$ center symmetry is left unbroken in this phase. Accordingly, we label this phase "confining".

For $M \geq 2$ the solution $\rho_i = \frac{1}{2\pi}$ will be a minimum of the action until we reach values of $(T;\mu_1,\mu_2)$ for which

$$0 = \det H_{ij} = \left| \frac{\partial^2 S_{\mathrm{MM}}}{\partial\rho_i^l\,\partial\rho_j^l} \right| \tag{4.3}$$

for any fixed l. When the temperature or the chemical potentials are raised above these critical values, the flat distribution becomes an unstable saddle point of the matrix model, and the model thus enters a new phase which we will discuss in the next section. For now we note that (4.3) defines a phase transition condition of the matrix model.

It will be convenient to express the Hessian matrix in terms of the variables $\xi_l \equiv 2V_{\mathrm{ad}}^l$ and $\eta_l \equiv V_{\mathrm{bi}}^l$. Note first that in the special case $M = 2$, due to the identification $i \simeq i+M = i+2$, the Hessian matrix obtained from (3.19) takes the form[12]

$$H = \begin{pmatrix} \xi_l & 2\eta_l \\ 2\eta_l & \xi_l \end{pmatrix} . \tag{4.4}$$

The determinant factorizes as $\det H = -4\left(\eta_l - \frac{1}{2}\xi_l\right)\left(\eta_l + \frac{1}{2}\xi_l\right)$. For $M \geq 3$ the Hessian matrix is a tridiagonal, periodically continued matrix:

$$H_{ij} = \begin{cases} \xi_l & \text{for } j = i \\ \eta_l & \text{for } j = i \pm 1 \end{cases} \tag{4.5}$$

where, as usual, we make the identifications $i \simeq i + M$ and $j \simeq j + M$. The determinant of H factorizes as follows[13]

$$\det \begin{pmatrix} \xi & \eta & & \eta \\ \eta & \xi & \ddots & \\ & \ddots & \ddots & \eta \\ \eta & & \eta & \xi \end{pmatrix} = \prod_{j=1}^{M} \left(\xi + 2\cos\left(\frac{2\pi j}{M}\right)\eta \right) . \tag{4.6}$$

Thus, the determinant of H vanishes on any of the lines $\xi_l + 2\cos\left(\frac{2\pi j}{M}\right)\eta_l = 0$ for $j = 1,\ldots,M$. To single out the physically relevant condition for the vanishing of $\det H$ we will first consider the case $M = 12$ to gain intuition. For $M = 12$ the determinant in particular factorizes as

$$\det H = -36\,\xi_l^2\left(\eta_l^2 - \xi_l^2\right)^2\left(\eta_l^2 - \frac{\xi_l^2}{4}\right)\left(\eta_l^2 - \frac{\xi_l^2}{3}\right)^2 \tag{4.7}$$

where l is fixed. In figure 2 we have divided the (ξ_l,η_l) plane into regions where H is positive-definite (denoted by $+$) and where H is indefinite (denoted by $-$).

[12] We omit here, and in the following, the overall factor of $\frac{N^2}{\pi}$ in eq. (3.19) for notational simplicity.

[13] This formula is a special case of (B.6).

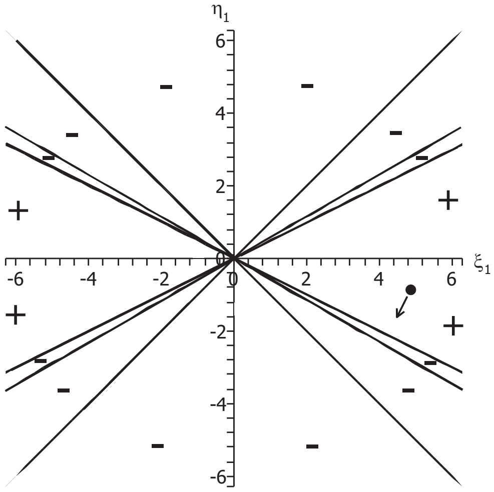

Figure 2: Regions of positive-definiteness and indefiniteness of H for the case $M = 12$. Regions where H is positive-definite (corresponding to a local minimum of S_{MM}) are marked by $+$; regions where H is indefinite (corresponding to an unstable saddle point of S_{MM}) are marked by $-$. The lines represent the locus of $\det H = 0$. The physically accessible region of the (ξ_l, η_l) plane is bounded from above by the ξ_l axis and from below by the line of the numerically smallest negative slope. This is illustrated by the dot which corresponds to $(T; \mu_1, \mu_2) = (0.1; 0.8, 0.8)$ and $l = 1$. The arrow shows how the dot will move as the temperature is increased, keeping μ_1 and μ_2 fixed.

Thus regions marked by $+$ correspond to a local extremum (minimum) of S_{MM}, and regions marked by $-$ correspond to unstable saddle points. In figure 2 we have furthermore marked the region occupied by the $\mathcal{N} = 2$ quiver gauge theory matrix model in the low temperature phase by plotting (ξ_1, η_1) for $(T; \mu_1, \mu_2) = (0.1; 0.8, 0.8)$. For fixed chemical potentials, $z^B_{\text{ad}}, z^F_{\text{ad}}, z^B_{\text{bi}}, z^F_{\text{bi}}$ all increase monotonically with the temperature. Therefore, as the temperature increases, the dot in figure 2 will move as indicated and hit the instability line $\eta_l = -\frac{1}{2}\xi_l$ at the phase transition temperature.

By the same analysis, for any $M \geq 2$ the phase transition occurs at the instability line $\eta_l = \alpha(M)\xi_l$ where $\alpha(M)$ is the numerically smallest negative slope of the zero lines of the Hessian determinant. For all $M \geq 3$ we find from (4.6) that $\alpha(M) = -\frac{1}{2}$ (corresponding to $j = M$). For $M = 2$ we also find $\alpha(M) = -\frac{1}{2}$. Indeed, note that for $M \geq 2$ the matrix obtained by substituting $\eta_l = -\frac{1}{2}\xi_l$ in eqs. (4.4) and (4.5) will have a zero eigenvalue (with $(1, 1, \ldots, 1)$ as an eigenvector) and hence zero determinant.

The large M limit As a consistency check, we can derive that $\lim_{M\to\infty} \alpha(M) = -\frac{1}{2}$ by a different route. We take the continuum limit $M \to \infty$ in the quiver direction. The quiver label i thus becomes a continuous angular parameter ϑ which we take to be 2π-periodic;

i.e., we identify $\vartheta \simeq \vartheta + 2\pi$. Accordingly we make the substitutions

$$(\rho_i^l)^2 \longrightarrow (\rho^l(\vartheta))^2 \tag{4.8}$$

$$(\rho_i^l \rho_{i+1}^l) \longrightarrow -\frac{1}{2}\big(\dot{\rho}^l(\vartheta)\big)^2 + (\rho^l(\vartheta))^2 \tag{4.9}$$

where $\dot{}$ denotes $\frac{d}{d\vartheta}$. The matrix model action (3.19) thus becomes[14]

$$S_{\rm MM}[\boldsymbol{\rho}] = \frac{N^2 M}{(2\pi)^2} \sum_{l=1}^{\infty} \int_0^{2\pi} d\vartheta \left[(\xi_l + 2\eta_l)(\rho^l(\vartheta))^2 - \eta_l \big(\dot{\rho}^l(\vartheta)\big)^2 \right] . \tag{4.10}$$

The Euler-Lagrange equations obtained from this action are those of a harmonic oscillator,

$$\eta_l \ddot{\rho}^l(\vartheta) + (\xi_l + 2\eta_l)\, \rho^l(\vartheta) = 0 \tag{4.11}$$

where $l = 1, 2, \ldots$. Note here that it is the bifundamental contribution in (3.19) that gives rise to the derivative term in (4.10) and in turn to the mass term for the harmonic oscillator. Thus, the harmonic oscillator EOM's in the large M limit is a pure 'quiver phenomenon'. Solutions to these equations will become unstable when the tension $\tau \equiv (\xi_l + 2\eta_l)$ goes from $\tau > 0$ to $\tau < 0$. Thus, for large M, the phase transition will occur when $\eta_l = -\frac{1}{2}\xi_l$, consistent with what we found above.

We now return to the phase transition condition $\eta_l = \alpha(M)\, \xi_l$. Since $z_{\rm ad}^B, z_{\rm ad}^F, z_{\rm bi}^B, z_{\rm bi}^F$ are all monotonically increasing as functions of x and $0 \leq x < 1$, the $l = 1$ condition is the strongest. Therefore, the phase transition condition for $M \geq 2$ is

$$\text{for } M \geq 2 : \big(z_{\rm ad}^B(x; y_1, y_2) + z_{\rm ad}^F(x; y_1, y_2)\big) + 2\big(z_{\rm bi}^B(x; y_1, y_2) + z_{\rm bi}^F(x; y_1, y_2)\big) = 1\,. \tag{4.12}$$

Finally, in the special case $M = 1$ we immediately obtain $V_{\rm ad}^l + V_{\rm bi}^l = 0$ from (4.1) due to the identification $i \simeq i + M = i + 1$. Putting $l = 1$, this is precisely the phase transition condition (4.12). We thus conclude that for any M the phase transition condition is

$$\big(z_{\rm ad}^B(x; y_1, y_2) + z_{\rm ad}^F(x; y_1, y_2)\big) + 2\big(z_{\rm bi}^B(x; y_1, y_2) + z_{\rm bi}^F(x; y_1, y_2)\big) = 1\,. \tag{4.13}$$

In figure 3 below we have plotted the curves in the (T, μ) plane obtained from this condition for the cases $(\mu_1, \mu_2) = (\mu, 0)$; $(\mu_1, \mu_2) = (0, \mu)$ and $(\mu_1, \mu_2) = (\mu, \mu)$. For each of these cases, the relevant curve defines the phase diagram of $\mathcal{N} = 2$ quiver gauge theory as a function of both temperature and chemical potential. Note that, as discussed in section 2.2, if one or both of the chemical potentials are larger than the inverse radius of the spatial manifold S^3, the theory develops tachyonic modes and becomes ill-defined. Therefore the line $\mu = 1/R$ defines a boundary of the phase diagram.

The phase transition condition (4.13) defines a phase transition temperature $T_H(\mu_1, \mu_2)$ as a function of the chemical potentials. We will refer to $T_H(\mu_1, \mu_2)$ as the Hagedorn temperature of $\mathcal{N} = 2$ quiver gauge theory. This terminology will be justified in section 4.2. We remark that the Hagedorn temperature at zero chemical potential is

$$T_H = -\frac{1}{\ln(7 - 4\sqrt{3})} \approx 0.37966 \tag{4.14}$$

[14]The extra prefactor $\frac{M}{2\pi}$ comes from changing the counting measure over i to the measure $d\vartheta$.

in units of R^{-1}, the inverse radius of the S^3. This is exactly the Hagedorn temperature for $\mathcal{N}=4$ SYM theory (cf. [6, 7]). The origin of this fact can be traced to the observation in [30] that in the large N limit the correlation functions of $\mathcal{N}=4$ U(N) SYM theory equal the corresponding correlation functions of the $\mathcal{N}=2$ quiver gauge theories obtained from orbifold projections. Since our computations rely on perturbation theory (namely, taking the $g \to 0$ limit of the action and then performing Gaussian path integrations), and we are furthermore taking the $N \to \infty$ limit, we should expect that the matrix model defined out of the quantum effective action will have the same behavior for the $\mathcal{N}=2$ quiver gauge theory as for the $\mathcal{N}=4$ SYM theory.

Furthermore, for small chemical potentials the Hagedorn temperature is given by

$$T_H(\mu_1,\mu_2) = \frac{1}{\beta_0} + c\big(\mu_1^2+2\mu_2^2\big) + c_{11}\mu_1^4 + c_{12}\mu_1^2\mu_2^2 + c_{22}\mu_2^4 + \mathcal{O}(\mu_i^6) \tag{4.15}$$

where the coefficients are

$$\beta_0 = -\ln(7-4\sqrt{3})\,, \quad c=-\frac{\sqrt{3}}{18}, \quad c_{11} = -\frac{\beta_0}{864}\left(\frac{362\beta_0-209\sqrt{3}\beta_0+2896\sqrt{3}-5016}{-627+362\sqrt{3}}\right) \tag{4.16}$$

$$c_{12}=\frac{\beta_0}{216}\left(\frac{1810\beta_0-1045\sqrt{3}\beta_0-2896\sqrt{3}+5016}{-627+362\sqrt{3}}\right), c_{22}=\frac{\beta_0}{108}\left(\frac{362\beta_0-209\sqrt{3}\beta_0-1448\sqrt{3}+2508}{-627+362\sqrt{3}}\right) \tag{4.17}$$

4.2 Solution above the Hagedorn temperature

As the temperature is increased beyond $T > T_H$, the attractive terms in the pairwise potential continue to increase in strength, and so the eigenvalues will become increasingly localized. The precise distribution can be determined, following [6], by the condition that a single additional eigenvalue α_i added on the i'th circle experiences no net force from the other eigenvalues on the circles $i-1$, i and $i+1$:

$$0 \;=\; \int_{-\pi}^{\pi} d\zeta\; 2V'_{\rm ad}(\alpha_i-\zeta)\,\rho_i(\zeta) \;+\; \int_{-\pi}^{\pi} d\zeta\; V'_{\rm bi}(\alpha_i-\zeta)\big(\rho_{i-1}(\zeta)+\rho_{i+1}(\zeta)\big) \tag{4.18}$$

where $V_{\rm ad}$ and $V_{\rm bi}$ are given in (3.12) and (3.13), respectively. This provides M equilibrium conditions for the lattice action

$$S_{\rm latt} \;=\; N\sum_{i=1}^{M}\sum_{l=1}^{\infty} \frac{a_l\rho_i^l \,+\, b_l\rho_{i-1}^l \,+\, b_l\rho_{i+1}^l}{l}\left(\mathrm{Tr}\,U_i^l + \mathrm{Tr}\,U_i^{-l}\right) \tag{4.19}$$

where

$$a_l \;=\; z^B_{\rm ad}(x^l;y_1^l,y_2^l) \;+\; (-1)^{l+1} z^F_{\rm ad}(x^l;y_1^l,y_2^l) \tag{4.20}$$

$$b_l \;=\; z^B_{\rm bi}(x^l;y_1^l,y_2^l) \;+\; (-1)^{l+1} z^F_{\rm bi}(x^l;y_1^l,y_2^l)\,. \tag{4.21}$$

The exact solution of (4.19) was found in [31]. It takes the form

$$\rho_i(\theta) = \frac{1}{\pi}\left(\sin^2\left(\frac{\theta_0^i}{2}\right) - \sin^2\left(\frac{\theta-\alpha_i}{2}\right)\right)^{1/2} \sum_{k=1}^{\infty} Q_k^i \cos\big((k-1/2)(\theta-\alpha_i)\big)\,,\;\; i=1\ldots M \tag{4.22}$$

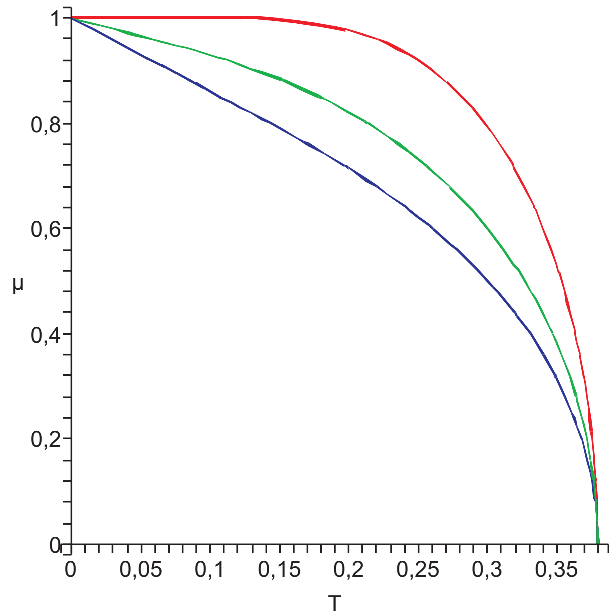

Figure 3: Phase diagram of $\mathcal{N}=2$ quiver gauge theory. The outermost curve is the transition line corresponding to $(\mu_1,\mu_2)=(\mu,0)$. It has slope 0 in the neighborhood of the point $(T,\mu)=(0,1)$. The inbetween curve corresponds to $(\mu_1,\mu_2)=(0,\mu)$, with slope $-\ln 2$ near $(0,1)$. The innermost curve corresponds to $(\mu_1,\mu_2)=(\mu,\mu)$, with slope $-\ln 4$ near $(0,1)$. The phase transition temperature at zero chemical potential is common for the three curves and equals $T=-\frac{1}{\ln(7-4\sqrt{3})}\approx 0.37966$ as in the $\mathcal{N}=4$ SYM case.

where

$$Q^i_k = 2\sum_{j=0}^{\infty}\left(a_{j+k}\rho_i^{j+k}+b_{j+k}\rho_{i-1}^{j+k}+b_{j+k}\rho_{i+1}^{j+k}\right)P_j(\cos\theta_0^i)\,. \tag{4.23}$$

The support of ρ_i is $[\alpha_i-\theta^i_0,\alpha_i+\theta^i_0]$. Here one must impose the consistency requirement

$$\rho_i^n = \int_{-\pi}^{\pi} d\theta\,\rho_i(\theta)\cos(n\theta)\,. \tag{4.24}$$

For simplicity the following analysis will be carried out only in the truncated case $a_{m>1}=b_{m>1}=0$ which shares the same qualitative behavior with the general case. For $n=1$, the consistency condition (4.24) then becomes

$$\begin{aligned}\rho_i^1 &= \frac{2}{\pi}\left(a_1\rho_i^1+b_1\rho_{i-1}^1+b_1\rho_{i+1}^1\right)\int_{-\theta_0^i}^{\theta_0^i} d\xi\left(\sin^2\left(\frac{\theta_0^i}{2}\right)-\sin^2\left(\frac{\xi}{2}\right)\right)^{1/2}\cos\left(\frac{\xi}{2}\right)\cos(\xi+\alpha_i)\\ &=\cos\alpha_i\left(a_1\rho_i^1+b_1\rho_{i-1}^1+b_1\rho_{i+1}^1\right)(2s_i^2-s_i^4)\end{aligned} \tag{4.25}$$

where $s_i^2\equiv\sin^2\left(\frac{\theta_0^i}{2}\right)$. In analogy with [6], θ^i_0 is determined from $Q^i_1=Q^i_0+2$, leading to the M equations

$$1 = 2s_i^2\left(a_1\rho_i^1+b_1\rho_{i-1}^1+b_1\rho_{i+1}^1\right)\,. \tag{4.26}$$

By means of (4.26), one can rewrite (4.25) as the set of M coupled equations

$$a_1(\rho_i^1)^2 + \big(b_1\rho_{i-1}^1 + b_1\rho_{i+1}^1 - a_1\cos\alpha_i\big)\rho_i^1 + \cos\alpha_i\left(\frac{1}{4} - b_1\rho_{i-1}^1 - b_1\rho_{i+1}^1\right) = 0\,. \qquad (4.27)$$

If one allows some of the α_i to be nonzero, one finds for any fixed μ_1, μ_2 a range of temperatures above $T_H(\mu_1, \mu_2)$ where (4.27) has no solution such that all s_i, given by (4.26), satisfy $0 \leq s_i^2 \leq 1$. Thus one must have $\alpha_1 = \cdots = \alpha_M = 0$.

With these centers of masses of the eigenvalue distributions, requiring that $0 \leq s_i^2 \leq 1$ leads to a unique solution of (4.27). This solution has all ρ_i^1 equal, as well as all s_i equal and given by

$$s_i^2 \;=\; \sin^2\left(\frac{\theta_0^i}{2}\right) \;=\; 1 - \sqrt{1 - \frac{1}{a_1 + 2b_1}}\,. \qquad (4.28)$$

With the assumption $a_{m>1} = b_{m>1} = 0$ the exact solution (4.22)–(4.23) thus truncates to

$$\rho_i(\theta) \;=\; \frac{1}{\pi s_i^2}\sqrt{s_i^2 - \sin^2\left(\frac{\theta}{2}\right)}\,\cos\left(\frac{\theta}{2}\right)\,. \qquad (4.29)$$

It is immediately clear from (4.28) that for temperatures above the Hagedorn temperature one has $\theta_0^i < \pi$; i.e., the eigenvalue distribution becomes gapped. In particular we note that the phase above the Hagedorn temperature has unbroken quiver translational invariance; i.e. $\rho_i = \rho_{i+1}$. The unbroken quiver translational invariance is expected on more general grounds due to the perturbative equivalence between $\mathcal{N} = 4$ SYM theory and $\mathcal{N} = 2$ quiver gauge theory [30], although it should be noted that in ref. [30] the gauge theories are studied on $\mathbb{R}^4$ rather than $S^1 \times S^3$.[15]

Free energy slightly above the Hagedorn temperature Using the Hagedorn temperature for small chemical potentials given in (4.15)–(4.17) we can compute the free energy slightly above the Hagedorn temperature in analogy with [9]. Defining $\Delta T \equiv T - T_H(\mu_1, \mu_2)$, we find for $0 < \Delta T \ll 1$ the perturbative expansion

$$\begin{aligned}\frac{F}{N^2 M} \;=\; &-\beta_0\frac{3}{8}\left(1 - \beta_0\frac{2\sqrt{3}+\beta_0}{36}\big(\mu_1^2 + 2\mu_2^2\big) + \mathcal{O}(\mu_i^4)\right)\Delta T \\ &-\beta_0^2\sqrt{\frac{3}{8}}\left(1 - \beta_0\frac{4+\sqrt{3}\beta_0}{24\sqrt{3}}\big(\mu_1^2 + 2\mu_2^2\big) + \mathcal{O}(\mu_i^4)\right)\Delta T^{3/2} + \mathcal{O}(\Delta T^2)\,. \qquad (4.30)\end{aligned}$$

High-temperature behavior of free energy In the $T \to \infty$ limit the pairwise attractive potentials grow to infinite strength, so the eigenvalues of the holonomy matrices U_i localize to extremely small intervals; i.e. the eigenvalue distribution functions will become delta functions, $\rho_i(\theta_i) \to \delta(\theta_i)$. (This is also clear from (4.28) since for $T \to \infty$ one has $a_1, b_1 \to \infty$ and thus $\theta_0 \to 0$. The normalization condition (3.14) then implies

[15] Note that non-perturbative effects could potentially destroy the quiver translational invariance. However, the computation carried out in this section is only valid perturbatively, so we would not expect to see such effects. See refs. [32 – 36] for work on non-perturbative equivalence between parent/daughter gauge theories related by orbifold and orientifold projections.

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$\rho_i(\theta_i) \to \delta(\theta_i)$.) Therefore $\rho_i^l \to 1$, and so from (3.19) and (3.6)–(3.9) we find that the free energy in the $T \to \infty$ limit is

$$F \;=\; -N^2 M \left(\frac{\pi^2 T^4}{6} + \frac{T^2}{4}\left(\mu_1^2 + 2\mu_2^2\right) - \frac{1}{32\pi^2}\left(\mu_1^4 - 4\mu_1^2\mu_2^2\right) \right) \mathrm{Vol}(S^3)\,. \tag{4.31}$$

Here we have applied the polylogarithm regularization procedure described in appendix E of ref. [37] in order to obtain (4.31).[16] We note that the free energy scales as $N^2 M$ as $N \to \infty$. This is to be expected from the orbifold projection invariant form of the fields (A.15)–(A.16) and (A.31)–(A.32), given that the free energy scales as N^2 for $\mathcal{N} = 4$ U(N) SYM theory for high temperatures in the $N \to \infty$ limit (cf. eq. (5.62) of [6]).

The fact that the free energies (4.30) and (4.31) are both of $\mathcal{O}(N^2 M)$ with respect to N suggests that the gauge theory in the phase above the Hagedorn temperature describes a non-interacting plasma of color non-singlet states. Furthermore, from the fact that the eigenvalue distribution (4.28)–(4.29) is gapped we can immediately conclude that the Polyakov loop $W(C)$ has non-zero expectation value as the trace does not average to zero in this case. In particular, this implies that the $\mathbb{Z}_N$ center symmetry is spontaneously broken in this phase. Accordingly, we label this phase "deconfined". Thus, we conclude that the phase transition defined by eq. (4.13) is a confinement/deconfinement phase transition. Since furthermore the derivative of the free energy with respect to the temperature is discontinuous at the phase transition temperature $T_H(\mu_1, \mu_2)$, we conclude that the transition is of first order. Furthermore, cf. [7, 6], we identify it with a Hagedorn phase transition, and $T_H(\mu_1, \mu_2)$ is thus the Hagedorn temperature of $\mathcal{N} = 2$ quiver gauge theory.

Twisted partition function In analogy with [38], one may study the twisted partition function for the quiver gauge theory, taking the boundary conditions for the spinor fields on the S^1 to be periodic rather than antiperiodic. In this case the Matsubara frequencies for the spinor fields will be the same as for the bosonic fields, and the twisted partition function $\widetilde{Z} = \mathrm{Tr}(-1)^F e^{-\beta H} = e^{-\widetilde{\Gamma}[U_i]}$ may be obtained directly from (3.5) by replacing $(-1)^{l+1} \longrightarrow (-1)$. Following [38] we choose to exhibit the $\mathbb{Z}_M$ symmetry of the (twisted) partition function by rewriting the adjoint and bifundamental holonomy factors in terms of eigenvectors under quiver node displacements $i \to i+1$. Indeed, define for $\omega \equiv e^{2\pi i/M}$,

$$\Omega_k^l \equiv \sum_{j=1}^{M} \omega^{-kj}\, U_j^l\,. \tag{4.32}$$

Under the quiver node displacement $U_i^l \longrightarrow U_{i+1}^l$ we find $\Omega_k^l \longrightarrow \omega^k \Omega_k^l$ so that Ω_k^l is an eigenvector under the displacement with the eigenvalue ω^k. Writing the holonomy factors

[16] Note that there is a minus sign missing on the right hand side of (E.4) for the $n \neq 1$ case.

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in terms of Ω_k^l, the twisted quantum effective action takes the form

$$\begin{aligned}\widetilde{\Gamma}[U_i] = -\frac{1}{M}\sum_{k=1}^{M}\sum_{l=1}^{\infty}\Bigg[& \frac{1}{l}\left(\frac{6x^{2l}-2x^{3l}}{(1-x^l)^3}\right)+\frac{1}{l}\left(\frac{x^l+x^{2l}}{(1-x^l)^3}\right)(y_1^l+y_1^{-l}) \\ & -\frac{1}{l}\left(\frac{2x^{3l/2}}{(1-x^l)^3}\right)(y_1^{l/2}+y_1^{-l/2})(y_2^l+y_2^{-l})\Bigg]\Big(\operatorname{Tr}\Omega_k^l\operatorname{Tr}\Omega_{-k}^{-l}\Big) \\ -\frac{1}{M}\sum_{k=1}^{M}\sum_{l=1}^{\infty}\Bigg[& \frac{1}{l}\left(\frac{x^l+x^{2l}}{(1-x^l)^3}\right)(y_2^l+y_2^{-l})-\frac{1}{l}\left(\frac{2x^{3l/2}}{(1-x^l)^3}\right)(y_1^{l/2}+y_1^{-l/2})\Bigg] \\ & \times\omega^{-k}\Big(\operatorname{Tr}\Omega_k^l\operatorname{Tr}\Omega_{-k}^{-l}+\operatorname{Tr}\Omega_k^{-l}\operatorname{Tr}\Omega_{-k}^{l}\Big)\,. \end{aligned}\tag{4.33}$$

It would be interesting to study the phase structure for the twisted partition function.

4.3 Quantum mechanical sectors

Since $\mathcal{N}=2$ quiver gauge theory is a conformal field theory, we can exploit the state/operator correspondence and map the Hamiltonian H to the dilatation operator D. As a consequence, the partition function of thermal $\mathcal{N}=2$ quiver gauge theory in the grand canonical ensemble takes the form

$$Z(T;\mu_1,\mu_2)=\operatorname{Tr}_{\mathcal{H}}\left(e^{-\beta D+\beta\mu_i Q_i}\right)\,. \tag{4.34}$$

Here the trace is taken over the entire Hilbert space $\mathcal{H}$ of gauge invariant operators. For weak 't Hooft coupling $\lambda\ll 1$, the dilatation operator D can be expanded perturbatively[17]

$$D=D_0+\sum_{n=2}^{\infty}\lambda^{n/2}D_n\,. \tag{4.35}$$

We let Q denote the total charge with respect to the Cartan generators of $\mathrm{SU}(2)_R\times\mathrm{U}(1)_R$, $Q=Q_1+Q_2$, with μ as the associated chemical potential.[18] Taking $\lambda=0$, the partition function (4.34) can be rewritten as

$$Z(T;\mu)=\operatorname{Tr}_{\mathcal{H}}\exp\Big(-\beta(D_0-Q)-\beta(1-\mu)Q\Big)\,. \tag{4.36}$$

Following [9], we now consider the region of small temperature and near-critical chemical potential

$$T\ll 1\,,\qquad\qquad 1-\mu\ll 1\,. \tag{4.37}$$

In this region, the Hilbert space of gauge invariant operators of $\mathcal{N}=2$ quiver gauge theory truncates to certain subsectors. To show this, first observe that in the region (4.37), operators with $D_0>Q$ appear with an extremely small weight factor in the partition function (4.36) since $\beta\gg 1$. On the other hand, for operators with $D_0=Q$, the weight factor

[17] This was shown for $\mathcal{N}=4$ $\mathrm{U}(N)$ SYM theory in [39, 40].

[18] Recall that in section 2.2 we defined the generator of the Cartan subalgebra of $\mathrm{SU}(2)_R$ to be σ_z rather than $\frac{1}{2}\sigma_z$ so that we have the associated charges Q_1,Q_2 implicitly given through eqs. (2.5)–(2.12). It is these charges we are referring to here, rather than the R-charges given in table 2.

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is non-negligible precisely because $1-\mu \ll 1$. Therefore, the partition function (4.36) is dominated by contributions from operators belonging to the subsector

$$\mathcal{H}_0 \equiv \left\{ \mathcal{O} \in \mathcal{H} \;\middle|\; (D_0 - Q)\mathcal{O} = 0 \right\} . \tag{4.38}$$

We thus conclude that by taking the near-critical limit

$$x \longrightarrow 0\,, \qquad xy \;\text{ fixed}\,, \tag{4.39}$$

the full Hilbert space $\mathcal{H}$ of gauge-invariant operators effectively truncates to the subsector $\mathcal{H}_0$. We will consider three concrete examples of this truncation below, obtained by either turning off one of the R-symmetry chemical potentials, or by putting them equal. As we remark below, the resulting subsectors are in a certain sense quantum mechanical.

Case 1: The 1/2 BPS sector We take $(\mu_1, \mu_2) = (\mu, 0)$, and thus the total Cartan charge is $Q = Q_1$. Taking the near-critical limit (4.39) of the partition function (3.10) then yields

$$Z(x;y) \;\longrightarrow\; \int \prod_{i=1}^{M} [\mathcal{D}U_i] \exp\left(\sum_{i=1}^{M} \sum_{l=1}^{\infty} \frac{(xy)^l}{l} \operatorname{Tr} U_i^l \operatorname{Tr} U_i^{-l} \right) . \tag{4.40}$$

Since the scalar field Φ_i has $D_0 = Q = 1$, we therefore conclude that the Hilbert space of gauge invariant operators truncates to the 1/2 BPS sector spanned by multi-trace operators of the form

$$\operatorname{Tr}\left(\Phi_{i_1}^{J_1}\right) \operatorname{Tr}\left(\Phi_{i_2}^{J_2}\right) \cdots \operatorname{Tr}\left(\Phi_{i_k}^{J_k}\right) . \tag{4.41}$$

It is clear that in the near-critical limit (4.39) all operators with covariant derivatives decouple. Thus all modes originating from defining a field theory on the spatial manifold S^3 are removed, and the locality of the field theory is lost. In this sense the resulting subsector of the field theory is quantum mechanical.

Case 2: The SU(2) sector We take $(\mu_1, \mu_2) = (0, \mu)$, and thus the total Cartan charge is $Q = Q_2$. Taking the near-critical limit (4.39) of the partition function (3.10) then yields

$$Z(x;y) \;\longrightarrow\; \int \prod_{i=1}^{M} [\mathcal{D}U_i] \exp\left(\sum_{i=1}^{M} \sum_{l=1}^{\infty} \frac{2(xy)^l}{l} \operatorname{Tr} U_i^l \operatorname{Tr} U_{i+1}^{-l} \right) . \tag{4.42}$$

Since the scalar fields $A_{i,(i+1)}$ and $B_{(i+1),i}$ both have $D_0 = Q = 1$, we therefore conclude that the Hilbert space of gauge invariant operators truncates to the SU(2) sector spanned by multi-trace operators of the form

$$\prod_{j=1}^{k} \operatorname{Tr}\left(Z^{(j)}_{1\rightarrow} Z^{(j)}_{2\rightarrow} \cdots Z^{(j)}_{J_j\rightarrow} \right) \tag{4.43}$$

where any letter $Z^{(j)}_{i_j\rightarrow}$ is one of the scalars $A_{i,(i+1)}$ or $B_{(i+1),i}$. The subscripts '$\longrightarrow$' denote that the quiver labels on the fields in question must trace out a closed loop on the quiver diagram in figure 1 so as to ensure gauge invariance. I.e., an example of a gauge invariant single-trace operator is $\operatorname{Tr}\left(A_{i,(i+1)} A_{(i+1),(i+2)} B_{(i+2),(i+1)} B_{(i+1),i}\right)$.

Case 3: The $\mathbf{SU(2|3)/\mathbb{Z}_M}$ sector We take $(\mu_1, \mu_2) = (\mu, \mu)$ and thus the total Cartan charge is $Q = Q_1 + Q_2$. Taking the near-critical limit (4.39) of the partition function (3.10) then yields

$$Z(x;y) \longrightarrow \int \prod_{i=1}^{M} [\mathcal{D}U_i] \exp\left[\sum_{i=1}^{M}\sum_{l=1}^{\infty}\left(\frac{(xy)^l + 2(-1)^{l+1}(xy)^{3l/2}}{l}\right)\operatorname{Tr}U_i^l \operatorname{Tr}U_i^{-l} \right.$$
$$\left. + \sum_{i=1}^{M}\sum_{l=1}^{\infty}\frac{2(xy)^l}{l}\operatorname{Tr}U_i^l \operatorname{Tr}U_{i+1}^{-l}\right]. \tag{4.44}$$

Since the scalar fields $A_{i,(i+1)}, B_{(i+1),i}, \Phi_i$ all have $D_0 = Q = 1$, and the Weyl spinor field $\overline{\psi_{\Phi,i}}$ has $D_0 = Q = \frac{3}{2}$, we therefore conclude that the Hilbert space of gauge invariant operators truncates to a subsector spanned by multi-trace operators of the form

$$\prod_{j=1}^{k} \operatorname{Tr}\left(W^{(j)}_{1\to}W^{(j)}_{2\to}\cdots W^{(j)}_{J_j\to}\right) \tag{4.45}$$

where any letter $W^{(j)}_{i_j\to}$ is either one of the scalars $A_{i,(i+1)}, B_{(i+1),i}, \Phi_i$, or the Weyl spinor $\overline{\psi_{\Phi,i}}$. Otherwise, the notation is as explained below (4.43).

It would be interesting to study this subsector further and determine its symmetry group. This group is presumably a subgroup of the SU(2|3) symmetry observed in the $\mathcal{N} = 4$ SYM case [9], and determined by the way the $\mathbb{Z}_M$ orbifolding breaks the embedding of SU(2|3) into the full $\mathcal{N} = 4$ superconformal group PSU(2, 2|4).

In [9] the authors considered weakly coupled $\mathcal{N} = 4$ $\mathrm{U}(N)$ SYM theory on $S^1 \times S^3$ with R-symmetry chemical potentials in similar near-critical regions of the phase diagram as studied here. It was found that the Hilbert space of gauge invariant operators truncates to similar subsectors as identified here, namely the 1/2 BPS sector, the SU(2) subsector or the SU(2|3) subsector, depending on which chemical potentials are turned on. Furthermore, the analysis in [9] was generalized to small, but non-zero 't Hooft coupling λ by utilizing the 1-loop correction D_2 to the dilatation operator (cf. the perturbative expansion (4.35)). In the large N limit, D_2 restricted to the SU(2) subsector becomes the Hamiltonian of an SU(2) spin chain; and restricted to the SU(2|3) subsector it becomes the Hamiltonian of an SU(2|3) spin chain. What is remarkable is that in both these cases, the spin chains are integrable [39 – 41], and that the truncated Hilbert spaces can be identified with subsectors of the complete dilatation operator of $\mathcal{N} = 4$ $\mathrm{U}(N)$ SYM theory that are expected to be closed to any order in perturbation theory.

For $\mathcal{N} = 2$ quiver gauge theory, the full dilatation operator along with possible integrable subsectors is not yet completely settled, so we are not able to immediately generalize our results to small, but non-zero 't Hooft coupling λ. However, we note that much progress has been made in this area. In particular, anomalous dimensions of various operators, the anomalous dimension matrix restricted to various subsectors, Bethe ansätze and integrability have been investigated in [42 – 53].

5. One-loop quantum effective action with scalar VEV's

In this section we will extend the matrix model for $\mathcal{N}=2$ quiver gauge theory on $S^1 \times S^3$ in section 3 to include non-zero VEV's for the scalar fields. To this end we calculate the quantum effective action at weak 't Hooft coupling to 1 loop in a slice of the configuration space of the background fields. To simplify the calculation we restrict to the case of zero R-symmetry chemical potentials. The potential we compute will be valid within the temperature range $0 \leq TR \ll \lambda^{-1/2}$. The origin of the bound $TR \ll \lambda^{-1/2}$ comes from the fact that R^{-1} provides a cutoff on the momentum integrals that appear in the loop diagrams that contribute to the effective action. Provided that R^{-1} is much larger than the inverse Debye length, one avoids infrared divergences which would require a resummation of the thermal mass of the fields.

The method employed for computing the effective potential will be the standard background field formalism. That is, we expand the quantum fields about classical background fields and path integrate over the fluctuations, discarding terms of cubic or higher order in the fluctuations. The background fields will be taken to be static and spatially homogeneous; thus, the potential obtained from the computation will be a static effective potential. Furthermore, we carry out the computation only in a slice of the configuration space in which the background fields are mutually "commuting" in a sense that conforms to the quiver structure.

We now proceed with a more detailed description of the calculation. For convenience we first rescale all the fields in the $\mathcal{N}=2$ quiver gauge theory Lagrangian density (as given in eqs. (A.34), (A.17), (A.18) and (A.33)) with a factor of g_{YM} as follows

$$\phi \longrightarrow \frac{1}{g_{\mathrm{YM}}}\phi\,. \tag{5.1}$$

We then expand the quantum fields about classical background fields by applying the following transformations to the Lagrangian density

$$\begin{aligned}
A_{i,(i+1)} &\longrightarrow A_{i,(i+1)} + a_{i,(i+1)} && (5.2)\\
B_{(i+1),i} &\longrightarrow B_{(i+1),i} + b_{(i+1),i} && (5.3)\\
\Phi_i &\longrightarrow \Phi_i + \phi_i && (5.4)\\
A_{\mu i} &\longrightarrow A_{\mu i} + \delta_{\mu 0}\,\alpha_i\,. && (5.5)
\end{aligned}$$

The background fields $a_{i,(i+1)}, b_{(i+1),i}, \phi_i$ and α_i are assumed to solve the Euler-Lagrange EOM's so that they are the VEV's of the corresponding fluctuating fields. We take the background fields to be static and spatially homogeneous, i.e. constant on $S^1 \times S^3$. This is to preserve the SO(4) isometry of S^3 as we will not examine the more exotic phases in which the vacuum spontaneously breaks rotational invariance.

The terms of the Lagrangian density arising after the transformations (5.2)–(5.5) are grouped by their order in the fluctuating fields. The terms of zeroth order are grouped into a tree-level Lagrangian density. The terms linear in the fluctuating fields combine to vanish as the background fields are solutions to the Euler-Lagrange EOM's. We discard terms containing fluctuating fields to cubic or higher order.[19] The quantum corrections to the tree-level Lagrangian density thus arise from path integrations over the terms quadratic in the fluctuations. The result will thus be valid to 1-loop order in the loop expansion.

It is technically difficult to compute the quantum corrections to the effective potential for arbitrary background fields. We will therefore only carry out the computation assuming that the background fields satisfy the constraints given below. These constraints are analogous to requiring that the background fields commute, while at the same time they respect the quiver structure of the theory.

First, the Polyakov loops must "commute" with the scalar VEV's:

$$\begin{aligned}
\alpha_i\, a_{i,(i+1)} - a_{i,(i+1)}\, \alpha_{i+1} &= 0\,, & \alpha_{i+1}\, \overline{a_{i,(i+1)}} - \overline{a_{i,(i+1)}}\, \alpha_i &= 0 \\
\alpha_{i+1}\, b_{(i+1),i} - b_{(i+1),i}\, \alpha_i &= 0\,, & \alpha_i\, \overline{b_{(i+1),i}} - \overline{b_{(i+1),i}}\, \alpha_{i+1} &= 0 \\
\left[\alpha_i, \phi_i\right] &= 0\,, & \left[\alpha_i, \overline{\phi_i}\right] &= 0\,.
\end{aligned} \tag{5.6}$$

Second, the scalar VEV's must "commute" among themselves:

$$\begin{aligned}
a_{i,(i+1)}\, \overline{a_{i,(i+1)}} - \overline{a_{(i-1),i}}\, a_{(i-1),i} &= 0\,, & b_{i,(i-1)}\, \overline{b_{i,(i-1)}} - \overline{b_{(i+1),i}}\, b_{(i+1),i} &= 0 \\
\left[\phi_i, \overline{\phi_i}\right] &= 0\,, & a_{(i-1),i}\, \overline{b_{(i+1),i}} - \overline{b_{i,(i-1)}}\, a_{i,(i+1)} &= 0 \\
\overline{a_{i,(i+1)}}\, b_{i,(i-1)} - b_{(i+1),i}\, \overline{a_{(i-1),i}} &= 0\,, & a_{i,(i+1)}\, \overline{\phi_{i+1}} - \overline{\phi_i}\, a_{i,(i+1)} &= 0 \\
\overline{a_{i,(i+1)}}\, \phi_i - \phi_{i+1}\, \overline{a_{i,(i+1)}} &= 0\,, & b_{(i+1),i}\, \overline{\phi_i} - \overline{\phi_{i+1}}\, b_{(i+1),i} &= 0 \\
\overline{b_{(i+1),i}}\, \phi_{i+1} - \phi_i\, \overline{b_{(i+1),i}} &= 0\,, & a_{i,(i+1)}\, b_{(i+1),i} - b_{i,(i-1)}\, a_{(i-1),i} &= 0 \\
\overline{a_{(i-1),i}}\, \overline{b_{i,(i-1)}} - \overline{b_{(i+1),i}}\, \overline{a_{i,(i+1)}} &= 0\,, & a_{i,(i+1)}\, \phi_{i+1} - \phi_i\, a_{i,(i+1)} &= 0 \\
\overline{a_{i,(i+1)}}\, \overline{\phi_i} - \overline{\phi_{i+1}}\, \overline{a_{i,(i+1)}} &= 0\,, & b_{(i+1),i}\, \phi_i - \phi_{i+1}\, b_{(i+1),i} &= 0 \\
\overline{b_{(i+1),i}}\, \overline{\phi_{i+1}} - \overline{\phi_i}\, \overline{b_{(i+1),i}} &= 0\,.
\end{aligned} \tag{5.7}$$

Since the zero modes $a_{i,(i+1)}, b_{(i+1),i}, \phi_i$ and α_i are constant over $S^1 \times S^3$, the tree-level action is obtained from the tree-level Lagrangian density by simply multiplying the volume of $S^1 \times S^3$. After imposing the constraints (5.6)–(5.7) the tree-level action takes the form

$$S^{(0)} = \frac{2\pi^2 \beta R}{g_{\rm YM}^2} \sum_{i=1}^{M} \mathrm{Tr}\left(a_{i,(i+1)}\, \overline{a_{i,(i+1)}} + b_{(i+1),i}\, \overline{b_{(i+1),i}} + \phi_i\, \overline{\phi_i}\right). \tag{5.8}$$

We choose an R_ξ gauge defined by adding the gauge fixing action

$$\begin{aligned}
S_{\rm g.f.} = \frac{1}{g_{\rm YM}^2} \frac{1}{2\xi} \sum_{i=1}^{M} \int d^4x \sqrt{|g|}\, \mathrm{Tr}\Bigg[\partial_\mu A_{\mu i} + i[\alpha_i, A_{0i}] + i\xi \bigg(\Big(\overline{a_{(i-1),i}} A_{(i-1),i} - A_{i,(i+1)} \overline{a_{i,(i+1)}}\Big) \\
+ \Big(a_{i,(i+1)}\, \overline{A_{i,(i+1)}} - \overline{A_{(i-1),i}}\, a_{(i-1),i}\Big) + \Big(\overline{b_{(i+1),i}}\, B_{(i+1),i} - B_{i,(i-1)}\, \overline{b_{i,(i-1)}}\Big) \\
+ \Big(b_{i,(i-1)}\, \overline{B_{i,(i-1)}} - \overline{B_{(i+1),i}}\, b_{(i+1),i}\Big) + \left[\overline{\phi_i}, \Phi_i\right] + \left[\phi_i, \overline{\Phi_i}\right]\bigg)\Bigg]^2.
\end{aligned} \tag{5.9}$$

[19] Note that with the redefinition of fields in (5.1), discarding terms of cubic or higher order in the fluctuations is the analog of taking the $g_{\rm YM} \longrightarrow 0$ limit in section 3.1.

We will furthermore choose the Feynman gauge $\xi = 1$ for convenience. The virtue of this gauge fixing action is that, using (5.6)–(5.7), it cancels terms appearing in the Lagrangian density after the transformations (5.2)–(5.5) that contain both gauge field and scalar field fluctuations. Thus, one can do the path integrations over the gauge field fluctuations and over the scalar field fluctuations separately.

Specification of the vacuum We will restrict to the case where all the zero modes $a_{i,(i+1)}, b_{(i+1),i}, \phi_i$ and α_i are taken to be diagonal $N \times N$ matrices.[20] The most general ansatz satisfying all the constraints (5.6)–(5.7) is given by

$$a_{i,(i+1)} = \mathrm{diag}\big(e^{i\theta_1^i}, \ldots, e^{i\theta_N^i}\big)\, a_{(i-1),i} \tag{5.10}$$

$$b_{(i+1),i} = \mathrm{diag}\big(e^{-i\theta_1^i}, \ldots, e^{-i\theta_N^i}\big)\, b_{i,(i-1)} \tag{5.11}$$

$$\phi_i = \phi_{i+1} \tag{5.12}$$

$$\alpha_i = \alpha_{i+1}\,. \tag{5.13}$$

If we furthermore require the vacuum to respect the gauge invariance and quiver translational invariance of the action along with the quiver M-periodicity, the most general form is

$$a_{i,(i+1)} = \omega^k\, a_{(i-1),i} \tag{5.14}$$

$$b_{(i+1),i} = \omega^{-k}\, b_{i,(i-1)} \tag{5.15}$$

$$\phi_i = \phi_{i+1} \tag{5.16}$$

$$\alpha_i = \alpha_{i+1} \tag{5.17}$$

where $\omega = e^{2\pi i/M}$ and $k \in \mathbb{Z}$. This is the vacuum we will adhere to in the computations throughout this section. We will find that the expression for the quantum effective action is independent of the value of k in (5.14)–(5.15).

5.1 Quantum corrections from bosonic fluctuations

There are radiative corrections to the tree-level potential coming from path integrations over the part of the action that is quadratic in the bosonic fluctuations. Below we present in a bilinear form the part of the action that is quadratic in the bosonic fluctuations, as it appears after being added to the gauge fixing action (5.9) and the Fadeev-Popov ghost action, and the constraints (5.6)–(5.7) have been imposed. The path integrals will then be Gaussian and can be evaluated easily.

First we introduce some notation. Define

$$\mathbf{A}_{\mu\, mn} \equiv \begin{pmatrix} (A_{\mu\, 1})_{mn} \\ \vdots \\ (A_{\mu\, M})_{mn} \end{pmatrix}, \qquad \mathbf{A}_{mn} \equiv \begin{pmatrix} (A_{1,2})_{mn} \\ \vdots \\ (A_{M,1})_{mn} \end{pmatrix}, \tag{5.18}$$

$$\mathbf{B}_{mn} \equiv \begin{pmatrix} (B_{1,M})_{mn} \\ \vdots \\ (B_{M,(M-1)})_{mn} \end{pmatrix}, \qquad \mathbf{\Phi}_{mn} \equiv \begin{pmatrix} (\Phi_1)_{mn} \\ \vdots \\ (\Phi_M)_{mn} \end{pmatrix} \tag{5.19}$$

[20] When the VEV's are allowed to be off-diagonal, satisfying the constraints (5.6)–(5.7) along with the quiver M-periodicity (i.e., $a_{(i+M),(i+M+1)} = a_{i,(i+1)}$ etc.) ultimately leads to relations between N and M, such as $N\,|\,M$.

so that, e.g.,

$$(\mathbf{A}^T)_{mn} = \Big((A_{1,2})_{mn},\, \ldots,\, (A_{M,1})_{mn}\Big) \qquad \text{and} \qquad \mathbf{A}^*_{mn} = \begin{pmatrix} (\overline{A_{1,2}})_{nm} \\ \vdots \\ (\overline{A_{M,1}})_{nm} \end{pmatrix} . \tag{5.20}$$

Furthermore, we define for fixed m, n the fluctuation operators $\Box_g^{mn}, \Box_{\mathbf{A}}^{mn}, \Box_{\mathbf{B}}^{mn}$ and $\Box_{\mathbf{\Phi}}^{mn}$ as certain $M \times M$ matrices (labelled by $i, j = 1, \ldots, M$) whose detailed form is given in (B.1)–(B.4). Then the part of the action that is quadratic in the bosonic fluctuations (including the Fadeev-Popov ghosts $\overline{c_i}, c_i$) can be written in the form ($k = 1, 2, 3$)

$$\begin{aligned} S_b = \frac{1}{g^2_{\mathrm{YM}}} \sum_{m,n=1}^{N} \int d^4x \sqrt{|g|} \,\Big(& \frac{1}{2} (\mathbf{A}_k^{\perp\, T})_{mn} \Box_g^{mn} (\mathbf{A}_k^{\perp})_{nm} + \frac{1}{2} (\partial_k \mathbf{F}^T)_{mn} \Box_g^{mn} (\partial_k \mathbf{F})_{nm} \\ & + \frac{1}{2} (\mathbf{A}_0^T)_{mn} \Box_g^{mn} \mathbf{A}_{0\,nm} + (\overline{\mathbf{c}}^T)_{mn} \Box_g^{mn} \mathbf{c}^*_{mn} \\ & + (\mathbf{A}^T)_{mn} \Box_{\mathbf{A}}^{mn} \mathbf{A}^*_{mn} + (\mathbf{B}^T)_{mn} \Box_{\mathbf{B}}^{mn} \mathbf{B}^*_{mn} + (\mathbf{\Phi}^T)_{mn} \Box_{\mathbf{\Phi}}^{mn} \mathbf{\Phi}^*_{mn} \Big) \end{aligned} \tag{5.21}$$

where, as in section 3.1, the spatial components of the gauge field have been decomposed into a transversal (i.e., divergenceless) part $(A_i^{\perp})^k$ and a longitudinal part $(\nabla F_i)^k$. Thereby all the fields have been written in terms of S^3 spherical harmonics. The path integrations over the bosonic fluctuations $A_{i,(i+1)}, B_{(i+1),i}, \Phi_i$ and $A_{\mu\, i}$ can now readily be done and yield the formal expression[21]

$$\begin{aligned} \Gamma_{\mathrm{bos}}\big[\alpha_i, a_{i,(i+1)}, b_{(i+1),i}, \phi_i\big] = & \frac{1}{2} \sum_{m,n=1}^{N} \mathrm{Tr} \ln \det \Box_g^{mn} + \sum_{m,n=1}^{N} \mathrm{Tr} \ln \det \Box_{\mathbf{A}}^{mn} \\ & + \sum_{m,n=1}^{N} \mathrm{Tr} \ln \det \Box_{\mathbf{B}}^{mn} + \sum_{m,n=1}^{N} \mathrm{Tr} \ln \det \Box_{\mathbf{\Phi}}^{mn} . \end{aligned} \tag{5.22}$$

Here the traces are taken over the Matsubara frequencies and over the S^3 spherical harmonics, and the determinants are taken over the i, j indices of the operators $\Box_g^{mn}, \Box_{\mathbf{A}}^{mn}, \Box_{\mathbf{B}}^{mn}$ and $\Box_{\mathbf{\Phi}}^{mn}$. Let us define here for convenience

$$\begin{aligned} v_{i,j;\,n,m} \equiv 2 \Big(& \big((a_{i,(i+1)})_{nn} - \omega^{-j} (a_{i,(i+1)})_{mm}\big) \big((\overline{a_{i,(i+1)}})_{nn} - \omega^{j} (\overline{a_{i,(i+1)}})_{mm}\big) \\ & + \big((b_{(i+1),i})_{nn} - \omega^{j} (b_{(i+1),i})_{mm}\big) \big((\overline{b_{(i+1),i}})_{nn} - \omega^{-j} (\overline{b_{(i+1),i}})_{mm}\big) \\ & + \big((\phi_i)_{nn} - (\phi_i)_{mm}\big) \big((\overline{\phi_i})_{nn} - (\overline{\phi_i})_{mm}\big) \Big) . \end{aligned} \tag{5.23}$$

Now we apply the determinant formula (B.6) to the formal expression (5.22) for Γ_{bos}. Then we take the traces over the Matsubara frequencies and over the S^3 spherical harmonics,

[21] We are using a rather sloppy notation here as the term involving $\Box_g^{mn}$ is to be interpreted as the total contribution from the path integrations over the transversal and longitudinal parts of the spatial components of the gauge field, the time component of the gauge field and the Fadeev-Popov ghosts. The individual contributions are explicitly written out in (5.24) below.

labelled by the angular momentum h (see table 1). This yields the following result

$$\begin{aligned}\Gamma_{\rm bos} &= \frac{1}{2M}\sum_{i,j=1}^{M}\sum_{m,n=1}^{N}\sum_{k=-\infty}^{\infty}\mathrm{Tr}_{h\geq 0}\ln\left[\left(\omega_k+(\alpha_i^{nn}-\alpha_i^{mm})\right)^2+\Delta_g^2+v_{i,j;\,n,m}\right]\\ &+\frac{1}{2M}\sum_{i,j=1}^{M}\sum_{m,n=1}^{N}\sum_{k=-\infty}^{\infty}\mathrm{Tr}_{h>0}\ln\left[\left(\omega_k+(\alpha_i^{nn}-\alpha_i^{mm})\right)^2+\Delta_s^2+v_{i,j;\,n,m}\right]\\ &+\left(\frac{1}{2}-1\right)\frac{1}{M}\sum_{i,j=1}^{M}\sum_{m,n=1}^{N}\sum_{k=-\infty}^{\infty}\mathrm{Tr}_{h\geq 0}\ln\left[\left(\omega_k+(\alpha_i^{nn}-\alpha_i^{mm})\right)^2+\Delta_s^2+v_{i,j;\,n,m}\right]\\ &+\frac{3}{M}\sum_{i,j=1}^{M}\sum_{m,n=1}^{N}\sum_{k=-\infty}^{\infty}\mathrm{Tr}_{h\geq 0}\ln\left[\left(\omega_k+(\alpha_i^{nn}-\alpha_i^{mm})\right)^2+\Delta_s^2+R^{-2}+v_{i,j;\,n,m}\right].\end{aligned}\tag{5.24}$$

Here the first line comes from the path integrations over the transverse part of the spatial gauge field, and the second line from the integrations over the longitudinal part. The third line comes from integrating over the temporal component of the gauge field and the Fadeev-Popov ghosts, contributing with the weights $\frac{1}{2}$ and -1, respectively. Finally, the fourth line comes from path integrating over the conformally coupled scalar fluctuations. Note that there is an exact cancellation between the contributions of all $h > 0$ spherical harmonics in the second and third line. As we will see in section 6, the surviving contribution from the $h = 0$ scalar spherical harmonic will be the dominating radiative correction in the low-temperature regime.

After performing the summations over the Matsubara frequencies and writing out the traces over the S^3 spherical harmonics with the appropriate eigenvalues of ∇^2 and their degeneracies (cf. table 1) we find

$$\begin{aligned}\Gamma_{\rm bos} = \frac{1}{2M}\sum_{i,j=1}^{M}\sum_{m,n=1}^{N}\Bigg[&-\beta\left(v_{i,j;\,n,m}\right)^{1/2}+2\sum_{l=1}^{\infty}\frac{1}{l}e^{-\beta l(v_{i,j;\,n,m})^{1/2}}\cos\left(\beta l(\alpha_i^{nn}-\alpha_i^{mm})\right)\\ &+\sum_{h=0}^{\infty}2h(h+2)\Bigg(\beta\left((h+1)^2R^{-2}+v_{i,j;\,n,m}\right)^{1/2}\\ &\qquad-2\sum_{l=1}^{\infty}\frac{1}{l}e^{-\beta l((h+1)^2R^{-2}+v_{i,j;\,n,m})^{1/2}}\cos\left(\beta l(\alpha_i^{nn}-\alpha_i^{mm})\right)\Bigg)\\ &+6\sum_{h=0}^{\infty}(h+1)^2\Bigg(\beta\left((h+1)^2R^{-2}+v_{i,j;\,n,m}\right)^{1/2}\\ &\qquad-2\sum_{l=1}^{\infty}\frac{1}{l}e^{-\beta l((h+1)^2R^{-2}+v_{i,j;\,n,m})^{1/2}}\cos\left(\beta l(\alpha_i^{nn}-\alpha_i^{mm})\right)\Bigg)\Bigg]\end{aligned}\tag{5.25}$$

where $v_{i,j;\,n,m}$ is defined in (5.23). This is the complete result for the contribution to the quantum effective action coming from bosonic fluctuations.

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5.2 Quantum corrections from fermionic fluctuations

The fluctuating fermionic fields will also give rise to radiative corrections that can be computed much along the lines of the bosonic corrections. It is convenient to carry out the calculation using $\mathcal{N}=4$ SYM notation for the Weyl spinor fields. The quiver structure of the action is taken into account by including appropriate factors Ω_c in the fluctuation operator as explained in appendix B.2.

The fermionic part of the Lagrangian density can be written in $\mathcal{N}=4$ SYM notation (cf. (A.25)) in the following bilinear form

$$\mathcal{L}_{\text{ferm}} = \frac{1}{g_{\text{YM}}^2} \sum_{i,j=1}^{M} \sum_{m,n=1}^{N} \left((\overline{\lambda_p})_{i;mn}, (\lambda_p)_{i;mn} \right) \mathbf{D}_{ij}^{mn} \begin{pmatrix} (\lambda_q)_{j;nm} \\ (\overline{\lambda_q})_{j;nm} \end{pmatrix} \tag{5.26}$$

where the fluctuation operator $\mathbf{D}_{ij}^{mn}$ is given in (B.11). Taking the determinant of $\mathbf{D}_{ij}^{mn}$ as explained in appendix B.2, and taking the traces over the fermionic Matsubara frequencies $\omega_k \equiv \frac{(2k+1)\pi}{\beta}$ and over the S^3 spherical harmonics, one finds the following result

$$\begin{aligned}\Gamma_{\text{ferm}} \;=\; & -\frac{4}{M} \sum_{i,j=1}^{M} \sum_{m,n=1}^{N} \sum_{h=0}^{\infty} h(h+1) \left(\beta \left(\left(h+\frac{1}{2} \right)^2 R^{-2} + v_{i,j;\,n,m} \right)^{1/2} \right. \\ & \left. + 2 \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l}\, e^{-\beta l \left(\left(h+\frac{1}{2}\right)^2 R^{-2} + v_{i,j;\,n,m} \right)^{1/2}} \cos\left(\beta l (\alpha_i^{nn} - \alpha_i^{mm}) \right) \right)\end{aligned} \tag{5.27}$$

where $v_{i,j;\,n,m}$ is defined in (5.23). The factor 4 comes from performing 4 path integrations. This is the complete result for the contribution to the quantum effective action coming from fermionic fluctuations.

We conclude that the quantum effective action of $\mathcal{N}=2$ quiver gauge theory with constant scalar field VEV's satisfying (5.6)–(5.7) is given by

$$\Gamma_{\text{eff}} \;=\; S^{(0)} + \Gamma_{\text{bos}} + \Gamma_{\text{ferm}} \tag{5.28}$$

where $S^{(0)}$ is the tree-level action

$$S^{(0)} = \frac{2\pi^2 \beta R}{g_{\text{YM}}^2} \sum_{i=1}^{M} \sum_{n=1}^{N} \left((a_{i,(i+1)})_{nn} (\overline{a_{i,(i+1)}})_{nn} + (b_{(i+1),i})_{nn} (\overline{b_{(i+1),i}})_{nn} + (\phi_i)_{nn} (\overline{\phi_i})_{nn} \right). \tag{5.29}$$

and Γ_{bos} and Γ_{ferm} are given in (5.25) and (5.27), respectively, with $v_{i,j;\,n,m}$ given in (5.23).

Note that the tree-level potential (5.29) is attractive, whereas the 1-loop quantum corrections in (5.25) and (5.27) are repulsive. As we will see in section 6, the competition between an attractive and a repulsive part of the potential will cause the equilibrium configurations of the eigenvalues of the scalar VEV's to be hypersurfaces.

5.3 Generalization to other $\mathbb{Z}_M$ orbifold field theories

The computations in this section and in appendix B can immediately be generalized to field theories obtained as $\mathbb{Z}_M$ projections of $\mathcal{N}=4$ U(NM) SYM theory where the action of $\mathbb{Z}_M$

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is that in (A.1) with ω replaced by ω^p for $p \in \mathbb{Z}$. For these theories,[22] the quantum fields must satisfy the $\mathbb{Z}_M$ invariance conditions obtained from (A.14) and (A.30) by replacing $\omega \to \omega^p$. In turn, the fields will take $\mathbb{Z}_M$ projection invariant forms analogous to (A.15)–(A.16) and (A.31)–(A.32), except that the bifundamental fields will have non-zero entries on the p'th super- or sub-diagonal. That is, A and B will have the non-zero entries $A_{i,(i+p)}$ and $B_{(i+p),i}$, respectively, and analogously for the respective superpartners χ_A and χ_B. As a result, the fluctuation operators $\Box_g^{mn}, \Box_{\mathbf{A}}^{mn}, \Box_{\mathbf{B}}^{mn}$ and $\Box_{\mathbf{\Phi}}^{mn}$ in (B.1)–(B.4) and Δ_{ij} in (B.18) will have non-zero entries on the p'th super- and sub-diagonals. Therefore, using the generalized determinant formula[23] (where $\omega \equiv e^{2\pi i/M}$)

$$\det \begin{pmatrix} z_1 & z_2 & z_3 & \cdots & z_M \\ z_M & z_1 & z_2 & \cdots & z_{M-1} \\ z_{M-1} & z_M & z_1 & \cdots & z_{M-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_2 & z_3 & z_4 & \cdots & z_1 \end{pmatrix} = \prod_{j=1}^{M} \left(z_1 + \omega^j z_2 + \omega^{2j} z_3 + \cdots + \omega^{(M-1)j} z_M \right) \tag{5.30}$$

we see that the fluctuation determinants factorize as in (B.6), with ω^{pj} replacing ω^j. We conclude that the quantum effective action of these more general $\mathbb{Z}_M$ orbifold field theories is given by the expression (5.28) where $S^{(0)}$ is given in (5.29) and Γ_{bos} and Γ_{ferm} are given in (5.25) and (5.27), respectively. The only change is that $v_{i,j;\,n,m}$ now takes the form

$$\begin{aligned} v_{i,j;\,n,m} \equiv 2 \Big(& \big((a_{i,(i+p)})_{nn} - \omega^{-pj} (a_{i,(i+p)})_{mm}\big)\big((\overline{a_{i,(i+p)}})_{nn} - \omega^{pj} (\overline{a_{i,(i+p)}})_{mm}\big) \\ & + \big((b_{(i+p),i})_{nn} - \omega^{pj} (b_{(i+p),i})_{mm}\big)\big((\overline{b_{(i+p),i}})_{nn} - \omega^{-pj} (\overline{b_{(i+p),i}})_{mm}\big) \\ & + \big((\phi_i)_{nn} - (\phi_i)_{mm}\big)\big((\overline{\phi_i})_{nn} - (\overline{\phi_i})_{mm}\big) \Big) \,. \end{aligned} \tag{5.31}$$

6. Topology transition and emergent spacetime

In this section we will find the solutions minimizing the effective potential computed in section 5 (given in (5.28), (5.29), (5.25), (5.27) and (5.23)) within the temperature range $0 \le TR \ll \lambda^{-1/2}$. We stress that, since the effective action of section 5 is only valid within a sector of constant background fields satisfying (5.6)–(5.7), the minima we find in this section are not the absolute minima of the gauge theory, and the phase transitions within this sector of background fields do not necessarily extend to phase transitions in the full gauge theory (cf. [14]). Nonetheless, we will see that the matrix model of section 5 exhibits some interesting dynamics.

The resulting distributions of eigenvalues will preserve the $\mathrm{SU}(2)\times\mathrm{U}(1)$ R-symmetry of $\mathcal{N}=2$ quiver gauge theory. As in ref. [13] we believe that due to the preserved R-symmetry, the minima found here are indeed the global minima of the effective action (within the sector of constant "commuting" VEV's). The key observation needed for obtaining the solutions is that both in the low-temperature regime and above the Hagedorn temperature

[22] These theories have also been considered in, e.g., refs. [25, 54, 30, 49].

[23] We emphasize that the entries of the $M \times M$ matrix in (5.30) are allowed to be complex numbers.

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T_H, the eigenvalue distributions for the scalar VEV's and the Polyakov loop can be solved for separately. As we will see, the Hagedorn transition causes a change in the topology of the joint eigenvalue distribution when the temperature is raised above T_H.

6.1 Low-temperature eigenvalue distribution

For temperatures low compared to the inverse radius of the S^3 (i.e., $TR \ll 1$), one can consistently discard terms in the quantum effective potential that are suppressed by Boltzmann factors,[24] and so one obtains the following low-temperature limit of the effective potential

$$\begin{aligned}\Gamma_{TR\ll 1} =& \frac{2\pi^2\beta R}{g_{\rm YM}^2}\sum_{i=1}^{M}\sum_{n=1}^{N}\Big((a_{i,(i+1)})_{nn}(\overline{a_{i,(i+1)}})_{nn} + (b_{(i+1),i})_{nn}(\overline{b_{(i+1),i}})_{nn} + (\phi_i)_{nn}(\overline{\phi_i})_{nn}\Big)\\ &-\frac{\beta}{2M}\sum_{i,j=1}^{M}\sum_{m,n=1}^{N}\Big[2\Big(\big((a_{i,(i+1)})_{nn}-\omega^{-j}\,(a_{i,(i+1)})_{mm}\big)\big((\overline{a_{i,(i+1)}})_{nn}-\omega^{j}\,(\overline{a_{i,(i+1)}})_{mm}\big)\\ &\qquad+\big((b_{(i+1),i})_{nn}-\omega^{j}\,(b_{(i+1),i})_{mm}\big)\big((\overline{b_{(i+1),i}})_{nn}-\omega^{-j}\,(\overline{b_{(i+1),i}})_{mm}\big)\\ &\qquad+\big((\phi_i)_{nn}-(\phi_i)_{mm}\big)\big((\overline{\phi_i})_{nn}-(\overline{\phi_i})_{mm}\big)\Big)\Big]^{1/2}. \end{aligned} \tag{6.1}$$

We observe that the eigenvalues of the Polyakov loop are not coupled to the eigenvalues of the scalar VEV's. Therefore, for low temperatures, the distribution of the Polyakov loop eigenvalues will be the same as in the case with zero scalar VEV's treated in section 4. Thus, we immediately conclude from section 4.1 that the eigenvalues $e^{i\alpha_i^{nn}}$ of the Polyakov loop (for i fixed) are uniformly distributed over S^1 for any temperature below the Hagedorn temperature. Note that for a uniform distribution of the angles α_i^{nn}, the terms multiplied by Boltzmann factors in (5.25) and (5.27) vanish exactly. Therefore we can consistently discard these terms as long as the temperature is below T_H.

In order to find the minima of (6.1) we make the observation that by making the identifications

$$a_{i,(i+1)} \cong \omega^{-1}a_{i,(i+1)} \tag{6.2}$$
$$b_{(i+1),i} \cong \omega\, b_{(i+1),i} \tag{6.3}$$
$$\phi_i \cong \phi_i \tag{6.4}$$

and applying them recursively to (6.1), the low-temperature effective potential reduces to

$$\begin{aligned}\Gamma_{TR\ll 1} =& \frac{2\pi^2\beta R}{g_{\rm YM}^2}\sum_{i=1}^{M}\sum_{n=1}^{N}\Big((a_{i,(i+1)})_{nn}(\overline{a_{i,(i+1)}})_{nn} + (b_{(i+1),i})_{nn}(\overline{b_{(i+1),i}})_{nn} + (\phi_i)_{nn}(\overline{\phi_i})_{nn}\Big)\\ &-\frac{\beta}{2}\sum_{i=1}^{M}\sum_{m,n=1}^{N}\Big[2\Big(\big((a_{i,(i+1)})_{nn}-(a_{i,(i+1)})_{mm}\big)\big((\overline{a_{i,(i+1)}})_{nn}-(\overline{a_{i,(i+1)}})_{mm}\big)\\ &\qquad+\big((b_{(i+1),i})_{nn}-(b_{(i+1),i})_{mm}\big)\big((\overline{b_{(i+1),i}})_{nn}-(\overline{b_{(i+1),i}})_{mm}\big)\\ &\qquad+\big((\phi_i)_{nn}-(\phi_i)_{mm}\big)\big((\overline{\phi_i})_{nn}-(\overline{\phi_i})_{mm}\big)\Big)\Big]^{1/2}. \end{aligned} \tag{6.5}$$

[24] We will verify a posteriori that this procedure is valid for all temperatures below the Hagedorn temperature.

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It is important to note that the identifications (6.2)–(6.4) correspond uniquely to the effective potential. That is, if one replaces ω by ω^q in (6.2)–(6.3), the potential (6.1) will not reduce to (6.5) for general M. To see this, note that, since all M powers of ω appear in (6.1), the order of ω^q must be M. Thus we must have $\gcd(q, M) = 1$ for all M which implies $q = 1$.

We now proceed with finding the minima of (6.5). These will be minima of (6.1) where the identifications (6.2)–(6.4) have been made. It is convenient to introduce the dimensionless variables

$$(\theta_i)_n \equiv \beta(\alpha_i)_{nn} \,, \tag{6.6}$$

$$(z_i)_{n,1} \equiv \beta(\phi_i)_{nn} \,, \qquad (z_i)_{n,2} \equiv \beta(a_{i,(i+1)})_{nn} \,, \qquad (z_i)_{n,3} \equiv \beta(b_{(i+1),i})_{nn} \tag{6.7}$$

and

$$(\boldsymbol{z}_i)_n \equiv \big((z_i)_{n,1}, (z_i)_{n,2}, (z_i)_{n,3}\big) \tag{6.8}$$

so that $(\boldsymbol{z}_i)_n \in \mathbb{C}^3$ for fixed i and n. Furthermore we introduce a norm on $\mathbb{C}^3$ defined by

$$\|\boldsymbol{w} - \boldsymbol{z}\| \equiv \left(\sum_{c=1}^{3} \big|(w_c) - (z_c)\big|^2 \right)^{1/2} \tag{6.9}$$

where $|\cdot|$ denotes the modulus. Written in this notation, (6.5) takes the form

$$\Gamma_{TR\ll 1} = \frac{2\pi^2 R}{g_{\mathrm{YM}}^2 \beta} \sum_{i=1}^{M} \sum_{n=1}^{N} \big\|(\boldsymbol{z}_i)_n\big\|^2 - \frac{1}{\sqrt{2}} \sum_{i=1}^{M} \sum_{m,n=1}^{N} \big\|(\boldsymbol{z}_i)_n - (\boldsymbol{z}_i)_m\big\| \,. \tag{6.10}$$

We will now take the continuum limit $N \to \infty$ and describe the eigenvalues of the Polyakov loop and the scalar VEV's by a joint eigenvalue distribution $\rho_i(\theta_i, \boldsymbol{z}_i)$ proportional to the density of eigenvalues at the point $(\theta_i, \boldsymbol{z}_i)$ (for some fixed i) and normalized as $\int d\theta_i\, d^3\boldsymbol{z}_i\, \rho_i(\theta_i, \boldsymbol{z}_i) = 1$. The continuum limit is obtained by applying the substitution

$$\frac{1}{N} \sum_{n=1}^{N} \big[\cdots\big] \longrightarrow \int d\theta_i\, d^3\boldsymbol{z}_i\, \rho_i(\theta_i, \boldsymbol{z}_i) \big[\cdots\big] \tag{6.11}$$

in analogy with (3.17). Here it is implied that the content of the brackets $\big[\cdots\big]$ carries an i label. In the continuum limit, the equation of motion for $\boldsymbol{z}_i$ obtained from (6.10) reads

$$\frac{\sqrt{2}\pi R}{\lambda \beta}\, \boldsymbol{z}_i = \int_{D_i} d^3\boldsymbol{z}'_i\, \rho_i(\boldsymbol{z}'_i) \frac{\boldsymbol{z}_i - \boldsymbol{z}'_i}{\|\boldsymbol{z}_i - \boldsymbol{z}'_i\|} \,. \tag{6.12}$$

Here $\rho_i(\cdot)$ is defined as the average $\rho_i(\boldsymbol{z}_i) \equiv \int_{-\pi}^{\pi} d\theta_i\, \rho_i(\theta_i, \boldsymbol{z}_i)$, and $D_i \subseteq \mathbb{C}^3$ denotes the support for ρ_i. The solution to (6.12) is given by the eigenvalue distribution

$$\rho_i(\boldsymbol{z}_i) = \frac{\delta(\|\boldsymbol{z}_i\| - r_i)}{2\pi^4 r_i^5} \tag{6.13}$$

where the radius r_i is given by

$$r_i = \frac{\lambda \beta}{\sqrt{2}\pi^3 R} \frac{1024}{945} \tag{6.14}$$

as can be checked straightforwardly. That is, (6.12) is satisfied for any $\boldsymbol{z}_i$ when the eigenvalues are distributed uniformly over an S^5 with the radius (6.14). Since (6.10) was obtained from the low-temperature effective potential (6.1) by making the orbifold identifications (6.2)–(6.4), we thus conclude that the minimum of (6.1) is a uniform distribution of the eigenvalues of the scalar VEV's over $S^5/\mathbb{Z}_M$ where the action of $\mathbb{Z}_M$ is precisely as in (A.1). This is consistent with [29], as one should expect in the low temperature limit where thermal effects are small. Since we found that the eigenvalues of the Polyakov loop are distributed uniformly over an S^1 for temperatures below the Hagedorn temperature, we conclude furthermore that the joint eigenvalue distribution of the scalar VEV's and the Polyakov loop is $S^5/\mathbb{Z}_M \times S^1$ in this temperature range.

It is remarkable that the eigenvalues of the scalar VEV's localize to a hypersurface in $\mathbb{C}^3$ rather than spreading out over the configuration space. The physical origin of the localization is essentially common for the matrix model developed here and the matrix model of [29], namely the competition between an attractive part of the quantum effective potential, and a repulsive part where the latter is generated by the path integrations. We interpret the eigenvalue distribution of the scalar VEV's as the emergence of the $S^5/\mathbb{Z}_M$ factor of the holographically dual $AdS_5 \times S^5/\mathbb{Z}_M$ string theory geometry. Finally we note that the hypersurface $S^5/\mathbb{Z}_M$ has the isometry group $\mathrm{SU}(2) \times \mathrm{U}(1)$, resulting from breaking the SU(4) isometry via the orbifold identifications (6.2)–(6.4). Since this is the full R-symmetry group $\mathrm{SU}(2)_R \times \mathrm{U}(1)_R$ of $\mathcal{N} = 2$ quiver gauge theory we believe (cf. [13]) that the minimum found here is indeed the global minimum of the effective action of section 5.

6.2 Eigenvalue distribution above the Hagedorn temperature

In the matrix model treated in sections 3 and 4 where the VEV's of the scalar fields were zero we observed that as the temperature is increased above $T_H \approx 0.38\, R^{-1}$, the Polyakov loop eigenvalue distributions open a gap. In this section we will examine how this phase transition manifests itself in the general case with non-zero scalar VEV's.

From the radius (6.14) one in particular finds that for low temperatures $\|\boldsymbol{z}_i\| \gg \lambda$, so that the tree-level term dominates over the quantum correction by a factor $\sim \frac{\|\boldsymbol{z}_i\|}{\lambda} \gg 1$. On the other hand, around the Hagedorn temperature T_H one finds $\|\boldsymbol{z}_i\| \sim \lambda$, and the tree-level term and the quantum corrections come within the same order of magnitude. It is therefore natural to re-express the effective potential in terms of the new variables

$$(\zeta_i)_{n,1} \equiv \lambda^{-1}(z_i)_{n,1}\,, \qquad (\zeta_i)_{n,2} \equiv \lambda^{-1}(z_i)_{n,2}\,, \qquad (\zeta_i)_{n,3} \equiv \lambda^{-1}(z_i)_{n,3}\,. \tag{6.15}$$

The computations in this section will be valid for temperatures in the range $0 \leq TR \ll \lambda^{-1/2}$. Since we can no longer neglect the terms multiplied by Boltzmann factors, we have to consider the full quantum effective action as computed in section 5 (given in (5.28), (5.29), (5.25), (5.27) and (5.23)). Once again, we apply the orbifold identifications (6.2)–(6.4), and express the result in terms of the variables $\theta_i, \boldsymbol{\zeta}_i$. However, the rescaling with the 't Hooft coupling λ in (6.15) will reorganize the perturbative expansion of the effective potential into

$$\Gamma_{\text{eff}} \;=\; \Gamma^{(0)}[\theta_i] + \lambda\,\Gamma^{(1)}[\theta_i, \boldsymbol{\zeta}_i] + \mathcal{O}(\lambda^2)\,. \tag{6.16}$$

Here the 0-loop term is

$$\Gamma^{(0)}[\theta_i] = \sum_{i=1}^{M}\sum_{m,n=1}^{N}\sum_{l=1}^{\infty}\frac{1}{l}\Big[1-\Big(z^B_{\rm ad}(e^{-\beta l R^{-1}};1,1)\,+\,2z^B_{\rm bi}(e^{-\beta l R^{-1}};1,1)\Big)$$
$$-(-1)^{l+1}\Big(z^F_{\rm ad}(e^{-\beta l R^{-1}};1,1)\,+\,2z^F_{\rm bi}(e^{-\beta l R^{-1}};1,1)\Big)\Big]\cos\big(l(\theta_i)_n - l(\theta_i)_m\big) \quad (6.17)$$

where $z^B_{\rm ad}, z^F_{\rm ad}, z^B_{\rm bi}, z^F_{\rm bi}$ are given in eqs. (3.6), (3.7), (3.8), (3.9), respectively, and $y_1 = y_2 = 1$ in this case since we are taking $\mu_1 = \mu_2 = 0$ here.

The 1-loop term in (6.16) is given by

$$\Gamma^{(1)}[\theta_i,\boldsymbol{\zeta}_i] = \frac{2\pi^2 RN}{\beta}\sum_{i=1}^{M}\sum_{n=1}^{N}\|(\boldsymbol{\zeta}_i)_n\|^2$$
$$-\frac{1}{\sqrt{2}}\sum_{i=1}^{M}\sum_{m,n=1}^{N}\|(\boldsymbol{\zeta}_i)_n-(\boldsymbol{\zeta}_i)_m\|\left(1+2\sum_{l=1}^{\infty}\cos\big(l(\theta_i)_n - l(\theta_i)_m\big)\right). \quad (6.18)$$

From the expansion (6.16) it is immediately obvious that to leading order in λ the θ_i are unaffected by the $\boldsymbol{\zeta}_i$. Therefore, to leading order, the eigenvalue distributions of the θ_i are the same as they were in the case with zero scalar VEV's treated in section 4. The eigenvalue distributions of the scalar VEV's can therefore be found by minimizing $\Gamma^{(1)}[\theta_i,\boldsymbol{\zeta}_i]$. Taking the large N limit of (6.18) according to (6.11) one finds

$$\frac{1}{N^2}\Gamma^{(1)} = \frac{2\pi^2 R}{\beta}\sum_{i=1}^{M}\int d\theta_i\, d^3\boldsymbol{\zeta}_i\,\rho_i(\theta_i,\boldsymbol{\zeta}_i)\,\|\boldsymbol{\zeta}_i\|^2$$
$$-\sqrt{2}\,\pi\sum_{i=1}^{M}\int d\theta_i\, d^3\boldsymbol{\zeta}_i\, d^3\boldsymbol{\zeta}'_i\,\rho_i(\theta_i,\boldsymbol{\zeta}_i)\rho_i(\theta_i,\boldsymbol{\zeta}'_i)\,\|\boldsymbol{\zeta}_i-\boldsymbol{\zeta}'_i\|\,. \quad (6.19)$$

Here we have used the identity $1+2\sum_{l=1}^{\infty}\cos\big(l(\theta_i)_n - l(\theta_i)_m\big) = 2\pi\delta\big((\theta_i)_n-(\theta_i)_m\big)$ which is simply the Fourier expansion of the delta function.

Now we proceed to minimize the action (6.19). Since the eigenvalue distributions for the Polyakov loop and the scalar VEV's can be solved for separately, the joint eigenvalue distribution factorizes:

$$\rho_i(\theta_i,\boldsymbol{\zeta}_i) = \frac{\rho_i(\theta_i)\,\delta\big(\|\boldsymbol{\zeta}_i\|-r_i(\theta_i)\big)}{\|\boldsymbol{\zeta}_i\|^5(1+(dr_i/d\theta_i)^2)^{1/2}\mathrm{Vol}(S^5)}\,. \quad (6.20)$$

Inserting (6.20) into the 1-loop term (6.19) one finds

$$\frac{1}{N^2}\Gamma^{(1)} = \frac{2\pi^2 R}{\beta}\sum_{i=1}^{M}\int d\theta_i\,\rho_i(\theta_i)\,r_i(\theta_i)^2 - 2\pi C\sum_{i=1}^{M}\int d\theta_i\,\rho_i(\theta_i)^2 r_i(\theta_i)$$
$$= \frac{2\pi^2 R}{\beta}\sum_{i=1}^{M}\int d\theta_i\left[\rho_i(\theta_i)\left(r_i(\theta_i)-\frac{C\beta}{2\pi R}\rho_i(\theta_i)\right)^2 - \frac{C^2\beta^2}{4\pi^2 R^2}\rho_i(\theta_i)^3\right] \quad (6.21)$$

where $C = \frac{2048\sqrt{2}}{945\pi}$. The final term only contributes to the 2-loop order distribution of the Polyakov loop eigenvalues and can therefore be ignored. Hence for a minimum we have

$$r_i(\theta_i) = \frac{C\beta}{2\pi R}\,\rho_i(\theta_i)\,. \tag{6.22}$$

As we know from section 4.2, when the temperature is raised above the Hagedorn temperature T_H, the Polyakov loop eigenvalue distribution becomes gapped and is thus an interval $[-\theta_0, \theta_0]$. The scalar VEV eigenvalues are now distributed uniformly over an $S^5/\mathbb{Z}_M$ fibered over this interval, with the radius of the $S^5/\mathbb{Z}_M$ at any point θ_i in the interval being proportional to the density of Polyakov loop eigenvalues at θ_i (for fixed TR). The $S^5/\mathbb{Z}_M$ thus shrinks to zero radius at the endpoints $\pm\theta_0$ of the interval: the topology of the joint eigenvalue distribution is an $S^6/\mathbb{Z}_M$ where the $\mathbb{Z}_M$ is understood to act on the S^5 transverse to an S^1 diameter. Thus, the Hagedorn phase transition manifests itself in the general case of non-zero scalar VEV's as a change in the topology of the joint eigenvalue distribution $S^5/\mathbb{Z}_M \times S^1 \longrightarrow S^6/\mathbb{Z}_M$.

In order to understand how the $S^6/\mathbb{Z}_M$ eigenvalue distribution may be realized in the dual AdS spacetime we first need to consider the S^1 part of the low-temperature distribution $S^1 \times S^5/\mathbb{Z}_M$. The eigenvalues of the Wilson line wound around the thermal circle give the positions of D2-branes[25] on the T-dual of the thermal circle in thermal AdS_5. As the temperature is raised higher and higher beyond T_H, the Polyakov loop eigenvalues become localized to smaller and smaller intervals. On the AdS side one therefore finds a localized D2-brane configuration. It was noted in [13] that a similar localization of D2-branes on a *spatial* circle, at finite temperature, was investigated in [55] where it was observed to produce a near-horizon geometry containing a non-contractible S^6. Moreover, it was predicted in [55] from supergravity that a $S^1 \times S^5 \to S^6$ topological transition of a Gregory-Laflamme type should take place. In the present case, where the dual spacetime is $AdS_5 \times S^5/\mathbb{Z}_M$, we expect the appearance of an $S^6/\mathbb{Z}_M$ in the near-horizon geometry of the localized configuration of D2-branes on the T-dual of the thermal circle.

We now address the important question regarding the stability of the saddle points (6.13) and (6.20) against off-diagonal fluctuations. As one may read off from (5.21), the mass of the ij entry of a fluctuating scalar field is $\sqrt{R^{-2} + \big((\varphi_a)_{ii} - (\varphi_a)_{jj}\big)^2}$ (for notational convenience we here use φ_a which are real-valued scalar fields related to the complex scalar fields by (A.9)–(A.10)). In the saddle point (6.13), one finds from (6.14) that

$$\big((\varphi_a)_{ii} - (\varphi_a)_{jj}\big)^2 \sim \frac{\lambda^2}{R^2} \tag{6.23}$$

and so the ratio of masses of an off-diagonal fluctuation to a diagonal fluctuation is $\frac{m_{\mathrm{off-diag}}}{m_{\mathrm{diag}}} \sim \sqrt{1 + \mathcal{O}(\lambda^2)}$. For small λ this ratio is very close to 1. A priori it thus appears possible for the fluctuating fields to have off-diagonal elements in this background, causing the background to be unstable.

Despite this we believe that the sector of constant and 'commuting' VEV's is interesting since it may have a connection with the dominant saddle points at strong 't Hooft coupling

[25] The D2-branes here are T-dual to the original D3-branes.

(where the assumption $[\varphi_a, \varphi_b] = 0$ seems natural). It is also worth remarking that at zero temperature and strong 't Hooft coupling one finds an $S^5/\mathbb{Z}_M$ distribution of the scalar VEV eigenvalues [29]. In a different vein, the topological phase transition $S^5/\mathbb{Z}_M \times S^1 \longrightarrow S^6/\mathbb{Z}_M$ provides a natural extension of the Hagedorn/deconfinement phase transition for the Polyakov loop eigenvalues studied in section 4, and it is a tantalizing question whether it extends to a phase transition of the full gauge theory at weak 't Hooft coupling.

Finally it should be noted that the effective action (5.28) has other saddle points in which off-diagonal fluctuations can have parametrically large masses. (Indeed, compare with [11] where these saddle points were associated with the Gregory-Laflamme instability in the gravity dual theory.) These saddle points are therefore guaranteed to be stable backgrounds. In particular, the effective action (5.28) has interesting physics beyond the saddle points studied in this section.

Generalization to other $\mathbb{Z}_M$ orbifold field theories The computations in this section immediately carry over to the more general $\mathbb{Z}_M$ orbifold field theories considered in section 5.3. In this paragraph we remark on the theory defined by letting the action of $\mathbb{Z}_M$ be that of (A.1) with ω replaced by ω^p for some fixed $p \in \mathbb{Z}$. The quantum effective action of the corresponding field theory is obtained from that of $\mathcal{N} = 2$ quiver gauge theory by defining $v_{i,j;\,n,m}$ to be as given in (5.31). The minima of this effective action are found by making the orbifold identifications

$$a_{i,(i+1)} \cong \omega^{-p} a_{i,(i+1)} \,, \qquad b_{(i+1),i} \cong \omega^{p}\, b_{(i+1),i} \,, \qquad \phi_i \cong \phi_i \,. \tag{6.24}$$

The resulting expression for the effective action is then precisely the same as in the case of $\mathcal{N} = 2$ quiver gauge theory treated in this section, and the conclusions carry directly over. In particular, having made the orbifold identifications (6.24), one finds the low-temperature joint eigenvalue distribution $S^5 \times S^1$ and the high-temperature distribution S^6. Alternatively, the joint eigenvalue distributions are $S^5/\mathbb{Z}_M \times S^1$ and $S^6/\mathbb{Z}_M$, respectively, where the action of $\mathbb{Z}_M$ is precisely the orbifold action defining the $\mathbb{Z}_M$ orbifold theory. It is important to note that the orbifold identifications (6.24) correspond uniquely to the quantum effective action of the field theory. Indeed, assume that we make the identifications (6.24) with some ω^q replacing ω^p. In order for the quantum effective action to reduce to an expression involving norms on $\mathbb{C}^3$ we must require ω^q to have the same order as ω^p. That is, we must have $\forall M \in \mathbb{N} : \gcd(q, M) = \gcd(p, M)$ which implies $q = p$. Identifying the above $S^5/\mathbb{Z}_M$ distribution with the $S^5/\mathbb{Z}_M$ part of the holographically dual $AdS_5 \times S^5/\mathbb{Z}_M$ spacetime, this shows in particular that, within this class of $\mathbb{Z}_M$ orbifold field theories, the geometry of the dual AdS spacetime is mirrored in the structure of the quantum effective action in a precise way.

7. Discussion and conclusions

In this paper we have investigated different aspects of the phase structure of $\mathcal{N} = 2$ $\mathrm{U}(N)^M$ quiver gauge theories. We have set up a matrix model for $\mathcal{N} = 2$ quiver gauge theories on $S^1 \times S^3$ with chemical potentials conjugate to the R-charges. We then found the stable saddle points of the model as a function of temperature and chemical potentials. More specifically, we identified a low and a high temperature phase separated by a threshold temperature $T_H(\mu)$ which marks a Hagedorn/deconfinement phase transition. The condition of stability of the low-temperature saddle point was translated into a phase diagram of $\mathcal{N}=2$ quiver gauge theory as a function of both temperature and chemical potentials. We observed that in regions of small temperature and near-critical chemical potential the Hilbert space of gauge invariant operators truncates to the SU(2) subsector, or to a larger subsector whose symmetry group has yet to be determined. More specifically, we found the SU(2) subsector when the chemical potential corresponding to the $\mathrm{SU}(2)_R$ factor of the R-symmetry group $\mathrm{SU}(2)_R\times\mathrm{U}(1)_R$ is turned on, whereas the larger subsector emerged from turning on both chemical potentials and setting them equal.

We then developed the matrix model of $\mathcal{N}=2$ quiver gauge theory in a different direction, allowing non-zero VEV's for the scalar fields, but setting the R-symmetry chemical potentials to zero. We did this by computing a 1-loop effective potential for constant and "commuting" VEV's, valid at weak 't Hooft coupling and in the temperature range $0\leq TR\ll\lambda^{-1/2}$. We furthermore obtained the effective potential for more general $\mathbb{Z}_M$ orbifold field theories by an immediate generalization. Then we found the equilibrium configurations of the eigenvalues of the Polyakov loop and the scalar VEV's. The eigenvalues of the scalar VEV's localize to a hypersurface in $\mathbb{C}^3$ due to a repulsive part of the effective potential of a Vandermonde type, originating from the quantum corrections. We found that at the Hagedorn temperature the topology of the joint distribution of the eigenvalues undergoes a phase transition $S^1\times S^5/\mathbb{Z}_M\to S^6/\mathbb{Z}_M$. Finally, we identified the $S^5/\mathbb{Z}_M$ part of the low-temperature eigenvalue distribution as the emergence of the $S^5/\mathbb{Z}_M$ part of the holographically dual geometry $AdS_5\times S^5/\mathbb{Z}_M$. It should be noted, though, that the latter is a dominant geometrical saddle at strong 't Hooft coupling while the "commuting" saddle found from the effective potential at weak 't Hooft coupling is not an absolute minimum [14, 13]. Extrapolating this identification to high temperatures, we furthermore note that the dual spacetime interpretation of the high-temperature $S^6/\mathbb{Z}_M$ is at present not entirely clear. We have also generalized the analysis to a class of $\mathbb{Z}_M$ orbifold field theories, thereby finding that the geometry of the dual AdS spacetime is similarly mirrored in the structure of the quantum effective action in a precise way.

There are several interesting future directions to pursue. It would be interesting to investigate other vacua of $\mathcal{N}=2$ quiver gauge theory which preserve less R-symmetry. In particular, such vacua could prove important when the matrix model with non-zero scalar VEV's developed in this paper is extended to include R-symmetry chemical potentials.

One could also consider the gravity duals of the phase transitions studied in this paper. In particular, the effective potential we computed in section 5 can be used to study the manifestation of Gregory-Laflamme instability from the weakly coupled gauge theory point of view [11]. This would proceed along the lines of [11] where, above a critical temperature $T_c\gg T_H$, the effective potential computed for $\mathcal{N}=4$ SYM theory was observed to develop new unstable directions along the scalar directions accompanied by new saddle points which only preserve an SO(5) subgroup of the global SO(6) isometry group. This phenomenon was identified as the weak coupling version of the Gregory-Laflamme localization instability

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of the small AdS_5 black hole in the gravity dual of the strongly coupled gauge theory.

Furthermore, the results obtained in this paper can be applied to compute the Polyakov-Maldacena loop [56] at weak coupling. It can also be computed at strong coupling following [57], so this is an interesting object to compare at weak and strong coupling.

It would be very interesting to study the subsectors of the Hilbert space of gauge invariant operators that we identified in section 4.3 in further detail, in particular to determine the symmetry group of the subsector corresponding to turning on both chemical potentials and setting them equal. A further point of particular interest would be to examine whether these subsectors are closed under the action of the full dilatation operator in analogy with [9]. More generally, these results could prove useful to further investigate the corresponding spin chain for the $\mathcal{N} = 2$ quiver gauge theory.

Another direction to pursue would be to examine, following [26, 58], whether the Hagedorn temperature of $\mathcal{N} = 2$ quiver gauge theory in the near-critical limit combined with the triple scaling limit of [46] can be matched to the Hagedorn temperature of Type IIB string theory on a pp-wave with a compact spacelike circle. This would involve computing the spectrum of a certain subsector of the SU(2) sector of gauge invariant operators which are dual to strings that wind about and/or have oscillators in the compact direction. This might require finding novel Bethe ansätze since the ground states of the spin chain governing the truncated SU(2) sector appear to be inherently different from, say, ferromagnetic ground states.

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A. Detailed description of $\mathcal{N} = 2$ quiver gauge theory

This appendix is intended to give a detailed description of $\mathcal{N} = 2$ quiver gauge theory, including details which the authors have not found elsewhere in the literature.

A.1 Relation to $\mathcal{N} = 4$ SYM theory

In this section we give a detailed description of how $\mathcal{N} = 2$ $\mathrm{U}(N)^M$ quiver gauge theory can be obtained by applying a $\mathbb{Z}_M$ projection to $\mathcal{N} = 4$ $\mathrm{U}(NM)$ SYM theory.

Consider Type IIB string theory and introduce a stack of NM coincident D3-branes into the 10-dimensional (initially flat) spacetime. It is well known that the low-energy effective field theory of open strings with endpoints attached to the D3-branes is 4-dimensional $\mathcal{N} = 4$ SYM theory with gauge group $\mathrm{U}(NM)$. The space transverse to the world volume

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of the D3-branes is $\mathbb{R}^6 \cong \mathbb{C}^3$ which has the isometry group SO(6). Now we consider the action of the subgroup $\mathbb{Z}_M$ on $\mathbb{C}^3$ given by

$$(z_1, z_2, z_3) \longrightarrow (z_1, \omega^{-1} z_2, \omega z_3)\,, \qquad \omega \equiv e^{2\pi i/M}\,. \tag{A.1}$$

The group $\mathbb{Z}_M$ is called the *orbifold group*. We will denote the resulting quotient of $\mathbb{C}^3$ by $\mathbb{C}^3/\mathbb{Z}_M$ where it is implied that the action of $\mathbb{Z}_M$ on $\mathbb{C}^3$ is always that given in eq. (A.1).

Consider now open strings living on the stack of D3-branes where the transverse space is $\mathbb{C}^3/\mathbb{Z}_M$. The low-energy effective field theory is no longer $\mathcal{N}=4$ $\mathrm{U}(NM)$ SYM theory. This is because associated with the orbifold group action (A.1) on the coordinates of $\mathbb{C}^3$ there is an orbifold group action on the scalar fields and their superpartners (to be defined below), and we must require that all quantum fields of $\mathcal{N}=4$ SYM theory be invariant under this action. The gauge theory obtained from $\mathcal{N}=4$ SYM theory by truncating the Hilbert spaces of quantum fields to $\mathbb{Z}_M$-invariant fields is called $\mathcal{N}=2$ *quiver gauge theory*.

The R-symmetry group of $\mathcal{N}=2$ quiver gauge theory is $\mathrm{SU}(2)_R \times \mathrm{U}(1)_R$. This is shown explicitly in appendix A.2 where the Lagrangian density of the quiver gauge theory is expressed in terms of $\mathrm{SU}(2)_R \times \mathrm{U}(1)_R$ invariants. The quiver gauge theory thus indeed has $\mathcal{N}=2$ supersymmetry.

The orbifold group action (A.1) breaks the gauge group $\mathrm{U}(NM)$ of the $\mathcal{N}=4$ theory into

$$\mathrm{U}(N)^{(1)} \times \mathrm{U}(N)^{(2)} \times \cdots \times \mathrm{U}(N)^{(M)} \tag{A.2}$$

which is thus the gauge group of $\mathcal{N}=2$ quiver gauge theory. We can see this as a manifestation of the fact that the quiver gauge theory is a low-energy effective field theory of open strings. Indeed, each of the M copies of $\mathbb{C}^3/\mathbb{Z}_M$ embedded in $\mathbb{C}^3$ will contain N coincident D3-branes, and an open string can attach its endpoints to any of the stacks. Finally, to conclude the enumeration of the symmetries of $\mathcal{N}=2$ quiver gauge theory, we note that it is known to be a conformally invariant theory like the parent $\mathcal{N}=4$ SYM theory [25].

In order to define the action of the orbifold group $\mathbb{Z}_M$ on the $\mathcal{N}=4$ SYM fields we first set up some notation. First $\mathbb{Z}_M$ is embedded into $\mathrm{U}(NM)$ by defining the twist matrix $\gamma \equiv \mathrm{diag}(1, \omega, \ldots, \omega^{M-1})$ and mapping $\mathbb{Z}_M \ni k \longmapsto \gamma^k \in \mathrm{U}(NM)$. (Note that the entries ω^j of γ are really $N \times N$ matrices.)[26] The action of $\mathbb{Z}_M$ on the $\mathcal{N}=4$ SYM fields is then

$$\phi \longrightarrow (\gamma^k)^\dagger\, (\rho \cdot \phi)\, \gamma^k \tag{A.3}$$

where $\rho \cdot \phi$ equals a phase times the field ϕ. For the scalar fields the phase is determined by their identifications with the z_1, z_2 and z_3 directions in $\mathbb{C}^3$ and comparing with (A.1). For the gauge field the phase is 1. For the spinor fields the phase equals that of their bosonic superpartner. Thus the condition for the $\mathcal{N}=4$ SYM fields ϕ to be invariant under the action of $\mathbb{Z}_M$ is

$$\phi = \gamma^\dagger\, (\rho \cdot \phi)\, \gamma\,. \tag{A.4}$$

[26] Note that this representation of $\mathbb{Z}_M$ satisfies $\mathrm{Tr}\,\gamma^k = 0$ for all $k \in \mathbb{Z}_M \setminus \{0\}$. As pointed out in ref. [54], this is needed for consistency (the cancellation of one-loop open string tadpole diagrams).

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In the following we will obtain the Lagrangian density of $\mathcal{N}=2$ $\mathrm{U}(N)^M$ quiver gauge theory by rewriting the $\mathcal{N}=4$ $\mathrm{U}(NM)$ SYM Lagrangian density and require that all the fields satisfy the $\mathbb{Z}_M$-invariance condition (A.4).

We now consider $\mathcal{N}=4$ $\mathrm{U}(NM)$ SYM theory on $\mathbb{R}\times S^3$ where the radius of S^3 is denoted by R. The scalar fields will couple conformally to the curvature of the S^3 through a quadratic term in the action. In the decompactification limit $R\to\infty$ this term will vanish. The action of $\mathcal{N}=4$ $\mathrm{U}(NM)$ SYM theory on $\mathbb{R}\times S^3$ equipped with a metric of Euclidean signature reads

$$\begin{aligned} S^{\mathcal{N}=4} = \int d^4x \ \mathrm{Tr}\Big(&\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{1}{2}(D_\mu\phi^i)(D_\mu\phi^i) + \frac{1}{2}R^{-2}\phi^i\phi^i - \frac{1}{4}g^2\left[\phi^i,\phi^j\right]\left[\phi^i,\phi^j\right] \\ &+ \frac{i}{2}\overline{\psi_p}\gamma_\mu D_\mu\psi_p - \frac{g}{2}\overline{\psi_p}\left[(\alpha^k_{pq}\phi^{2k-1} + i\beta^k_{pq}\gamma_5\phi^{2k}),\psi_q\right]\Big) \end{aligned} \tag{A.5}$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+ig[A_\mu,A_\nu]$ and $D_\mu=\partial_\mu+ig[A_\mu,\cdot\,]$. The traces are taken over the gauge indices. The indices have the ranges $\mu,\nu=0,\dots,3$; $i,j=1,\dots,6$; $p,q=1,\dots,4$ and $k=1,\dots,3$. Here ϕ^i are six real scalar fields and ψ_p are four 4-component Majorana spinors. Moreover, γ_μ are the 4-dimensional 4×4 gamma matrices and α^k and β^k are 4×4 matrices satisfying the relations

$$\{\alpha^k,\alpha^l\} = -2\delta^{kl}\mathbf{1}_4\,, \qquad \{\beta^k,\beta^l\} = -2\delta^{kl}\mathbf{1}_4\,, \qquad [\alpha^k,\beta^l]=0\,. \tag{A.6}$$

Explicit representations can be given as

$$\alpha^1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \qquad \alpha^2 = \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \qquad \alpha^3 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}, \tag{A.7}$$

$$\beta^1 = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}, \qquad \beta^2 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{pmatrix}, \qquad \beta^3 = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}. \tag{A.8}$$

The bosonic part of the quiver action To put the action of $\mathcal{N}=4$ SYM theory in a form suitable for performing the orbifold projection we now define three complex scalar fields

$$A=\frac{1}{\sqrt{2}}(\phi^1+i\phi^2)\,, \qquad B=\frac{1}{\sqrt{2}}(\phi^3+i\phi^4)\,, \qquad \Phi=\frac{1}{\sqrt{2}}(\phi^5+i\phi^6)\,. \tag{A.9}$$

The fields ϕ^i are Hermitian (since they transform in the adjoint representation of the gauge group $\mathrm{U}(NM)$), so by Hermitian conjugation of (A.9) we find

$$\overline{A}=\frac{1}{\sqrt{2}}(\phi^1-i\phi^2)\,, \qquad \overline{B}=\frac{1}{\sqrt{2}}(\phi^3-i\phi^4)\,, \qquad \overline{\Phi}=\frac{1}{\sqrt{2}}(\phi^5-i\phi^6)\,. \tag{A.10}$$

The scalar field part of the $\mathcal{N}=4$ SYM Lagrangian density written in terms of these fields takes the form

$$\begin{aligned} \mathcal{L}^{\mathcal{N}=4}_{\text{scalar}} &= \mathrm{Tr}\left(\frac{1}{2}(D_\mu\phi^i)(D_\mu\phi^i) + \frac{1}{2}R^{-2}\phi^i\phi^i - \frac{1}{4}g^2\left[\phi^i,\phi^j\right]\left[\phi^i,\phi^j\right]\right) \\ &= \mathrm{Tr}\left(D_\mu A\, D_\mu\overline{A} + D_\mu\overline{B}\, D_\mu B + D_\mu\Phi\, D_\mu\overline{\Phi}\right) \\ &\quad + R^{-2}\,\mathrm{Tr}\left(A\overline{A} + \overline{B}B + \Phi\overline{\Phi}\right) + \mathcal{L}^{\mathcal{N}=4}_D + \mathcal{L}^{\mathcal{N}=4}_F \end{aligned} \tag{A.11}$$

where the D and F terms are, respectively,

$$\mathcal{L}_D^{\mathcal{N}=4} = \frac{1}{2} g^2 \operatorname{Tr} \Big([A,\overline{A}] + [B,\overline{B}] + [\Phi,\overline{\Phi}] \Big)^2 \tag{A.12}$$

$$\mathcal{L}_F^{\mathcal{N}=4} = -2g^2 \operatorname{Tr} \Big([A,B]\,[\overline{A},\overline{B}] + [A,\Phi]\,[\overline{A},\overline{\Phi}] + [B,\Phi]\,[\overline{B},\overline{\Phi}] \Big)\,. \tag{A.13}$$

The scalar fields Φ, A and B can be identified with the z_1, z_2 and z_3 directions of the $\mathbb{C}^3$ (because they are the Goldstone bosons associated with breaking the translational invariance in the directions transverse to the D3-branes), so we have the orbifold group action $\rho : (\Phi, A, B) \mapsto (\Phi, \omega^{-1} A, \omega B)$, and the condition for these fields to be invariant under the $\mathbb{Z}_M$-transformation is then

$$\gamma^\dagger \Phi \gamma = \Phi\,, \qquad \gamma^\dagger A \gamma = \omega A\,, \qquad \gamma^\dagger B \gamma = \omega^{-1} B\,. \tag{A.14}$$

One easily checks that these conditions are satisfied by splitting the $NM \times NM$ matrix fields of the $\mathcal{N} = 4$ $\mathrm{U}(NM)$ SYM theory up into $M \times M$ block matrices whose entries are $N \times N$ matrices:

$$A_\mu = \begin{pmatrix} A_{\mu 1} & & & \\ & A_{\mu 2} & & \\ & & \ddots & \\ & & & A_{\mu M} \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & A_{1,2} & & & \\ & 0 & A_{2,3} & & \\ & & \ddots & \ddots & \\ & & & 0 & A_{(M-1),M} \\ A_{M,1} & & & & 0 \end{pmatrix}, \tag{A.15}$$

$$B = \begin{pmatrix} 0 & & & & B_{1,M} \\ B_{2,1} & 0 & & & \\ & B_{3,2} & \ddots & & \\ & & \ddots & 0 & \\ & & & B_{M,(M-1)} & 0 \end{pmatrix}, \qquad \Phi = \begin{pmatrix} \Phi_1 & & & \\ & \Phi_2 & & \\ & & \ddots & \\ & & & \Phi_M \end{pmatrix}. \tag{A.16}$$

Here $A_{\mu i}, A_{i,(i+1)}, B_{(i+1),i}$ and Φ_i are $N \times N$ matrices (where $i = 1, \ldots, M$ and we identify $i \simeq i+M$). Inserting the $\mathbb{Z}_M$-invariant forms of A_μ, A, B and Φ given in eqs. (A.15)–(A.16) into (A.11)–(A.13) the scalar field part of the $\mathcal{N} = 2$ quiver gauge theory Lagrangian density reads

$$\begin{aligned} \mathcal{L}_{\text{scalar}} = \sum_{i=1}^{M} \Big\{ & \operatorname{Tr} \Big[\Big(\partial_\mu A_{i,(i+1)} + ig A_{\mu i} A_{i,(i+1)} - ig A_{i,(i+1)} A_{\mu(i+1)} \Big) \\ & \qquad \times \Big(\partial_\mu \overline{A_{i,(i+1)}} + ig A_{\mu(i+1)} \overline{A_{i,(i+1)}} - ig \overline{A_{i,(i+1)}} A_{\mu i} \Big) \Big] \\ & + \operatorname{Tr} \Big[\Big(\partial_\mu B_{(i+1),i} + ig A_{\mu(i+1)} B_{(i+1),i} - ig B_{(i+1),i} A_{\mu i} \Big) \end{aligned}$$

$$
\begin{aligned}
&\times \Big(\partial_\mu \overline{B_{(i+1),i}} + ig A_{\mu i}\overline{B_{(i+1),i}} - ig\overline{B_{(i+1),i}} A_{\mu(i+1)}\Big)\Big] \\
&+ \operatorname{Tr}\Big[\Big(\partial_\mu \Phi_i + ig[A_{\mu i}, \Phi_i]\Big)\Big(\partial_\mu \overline{\Phi_i} + ig[A_{\mu i}, \overline{\Phi_i}]\Big)\Big] \\
&+ R^{-2}\operatorname{Tr}\Big(A_{i,(i+1)}\overline{A_{i,(i+1)}} + \overline{B_{(i+1),i}} B_{(i+1),i} + \Phi_i \overline{\Phi_i}\Big) \\
&+ \frac{1}{2} g^2 \operatorname{Tr}\Big[\Big(A_{i,(i+1)}\overline{A_{i,(i+1)}} - \overline{A_{(i-1),i}} A_{(i-1),i} \\
&\qquad + B_{i,(i-1)}\overline{B_{i,(i-1)}} - \overline{B_{(i+1),i}} B_{(i+1),i} + [\Phi_i, \overline{\Phi_i}]\Big)^2\Big] \\
&- 2g^2 \operatorname{Tr}\Big[\Big(A_{i,(i+1)} B_{(i+1),i} - B_{i,(i-1)} A_{(i-1),i}\Big) \\
&\qquad \times \Big(\overline{A_{(i-1),i}}\ \overline{B_{i,(i-1)}} - \overline{B_{(i+1),i}}\ \overline{A_{i,(i+1)}}\Big)\Big] \\
&- 2g^2 \operatorname{Tr}\Big[\Big(A_{i,(i+1)}\Phi_{i+1} - \Phi_i A_{i,(i+1)}\Big)\Big(\overline{A_{i,(i+1)}}\ \overline{\Phi_i} - \overline{\Phi_{i+1}}\ \overline{A_{i,(i+1)}}\Big)\Big] \\
&- 2g^2 \operatorname{Tr}\Big[\Big(B_{(i+1),i}\Phi_i - \Phi_{i+1} B_{(i+1),i}\Big)\Big(\overline{B_{(i+1),i}}\ \overline{\Phi_{i+1}} - \overline{\Phi_i}\ \overline{B_{(i+1),i}}\Big)\Big]\Bigg\}
\end{aligned}
\tag{A.17}
$$

Inserting the form of A_μ given in (A.15) into (A.5), the gauge field part of the $\mathcal{N} = 2$ quiver gauge theory Lagrangian density reads

$$
\mathcal{L}_{\text{gauge}} = \frac{1}{4}\sum_{i=1}^{M} \operatorname{Tr} F^i_{\mu\nu} F^i_{\mu\nu} \tag{A.18}
$$

where of course $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + ig[A^i_\mu, A^i_\nu]$.

The fermionic part of the quiver action The fermionic part of the $\mathcal{N} = 4$ SYM Lagrangian density reads

$$
\mathcal{L}^{\mathcal{N}=4}_{\text{ferm}} = \operatorname{Tr}\left(\frac{i}{2}\overline{\psi_p}\gamma_\mu D_\mu \psi_p - \frac{g}{2}\overline{\psi_p}\big[(\alpha^k_{pq}\phi^{2k-1} + i\beta^k_{pq}\gamma_5\phi^{2k}), \psi_q\big]\right) \tag{A.19}
$$

where the gamma matrices are given by

$$
\gamma_\mu \equiv \begin{pmatrix} 0 & \tau_\mu \\ \overline{\tau}_\mu & 0 \end{pmatrix}, \qquad \gamma_5 \equiv \gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \tag{A.20}
$$

$$
\tau_\mu \equiv (1, i\boldsymbol{\sigma}), \qquad \overline{\tau}_\mu \equiv (1, -i\boldsymbol{\sigma}) \tag{A.21}
$$

and representations of α^k and β^k are given in eqs. (A.7) and (A.8), respectively. The fields ψ_p, $p = 1, \ldots, 4$ are 4-component Majorana spinors which can be decomposed in terms of 2-component Weyl spinors as follows

$$
(\psi_p)^a \equiv \begin{pmatrix} (\lambda_p)_\alpha \\ (\overline{\lambda_p})^{\dot\alpha} \end{pmatrix}, \qquad (\overline{\psi_p})_a \equiv \begin{pmatrix} (\lambda_p)^\alpha \\ (\overline{\lambda_p})_{\dot\alpha} \end{pmatrix} \tag{A.22}
$$

where $a = 1, \ldots, 4$ is the spinor index on ψ_p. The Majorana spinors are related to their conjugates through the Majorana condition

$$
\psi_p = C\overline{\psi_p} \tag{A.23}
$$

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where the Majorana conjugation matrix is $C = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot\alpha\dot\beta} \end{pmatrix}$ with $\epsilon_{12} = -\epsilon_{21} = -1$.

Combining eqs. (A.22) and (A.20)–(A.21) one finds

$$\frac{1}{2}\overline{\psi_p}\gamma_\mu D_\mu \psi_p = (\lambda_p)^\alpha (\tau_\mu)_{\alpha\dot\beta} \overset{\leftrightarrow}{D}_\mu \, (\overline{\lambda_p})^{\dot\beta} \, . \tag{A.24}$$

Here the operator $\overset{\leftrightarrow}{D}_\mu$ is defined by $\chi_p \overset{\leftrightarrow}{D}_\mu \chi_q \equiv \frac{1}{2}\big(\chi_p D_\mu \chi_q - (D_\mu \chi_p)\, \chi_q\big)$.

It will be useful for exhibiting the R-symmetry of the quiver gauge theory to express the fermionic Lagrangian density in terms of the following Weyl spinors

$$\chi_A \equiv \overline{\lambda_1}\, , \qquad \chi_B \equiv \overline{\lambda_2}\, , \qquad \psi \equiv \overline{\lambda_3}\, , \qquad \psi_\Phi \equiv \lambda_4 \, . \tag{A.25}$$

Here $\chi_A, \chi_B, \psi, \psi_\Phi$ are the respective superpartners of A, B, A_μ, Φ. Note here that the bar used over the spinors in (A.25) is understood to mean the *Hermitian conjugate* whereas the bar over the λ_p in (A.24) denotes the usual conjugate of Weyl spinors. Explicitly, letting $\alpha = 1, 2$ be the spinor index and letting m, n be the gauge indices,

$$(\lambda_1)_{\alpha,mn} \equiv (\chi_A)^*_{\alpha,nm} = (\overline{\chi_A})_{\alpha,mn} \tag{A.26}$$

and

$$(\overline{\lambda_1})_{\dot\alpha,mn} = \big((\lambda_1)^*_{\alpha,mn}\big)^T = (\lambda_1)^*_{\alpha,nm} = (\chi_A)_{\alpha,mn} \tag{A.27}$$

and analogously for χ_B, ψ_Φ and ψ. In particular, note that all the Weyl spinors $\chi_A, \chi_B, \psi_\Phi$ and ψ have undotted indices.

Inserting the definitions (A.25) into the decomposition (A.24) we can write the kinetic part of the fermionic $\mathcal{N}=4$ SYM Lagrangian density (A.19) in the form

$$\begin{aligned} \mathcal{L}^{\mathcal{N}=4,\mathrm{kin}}_{\mathrm{ferm}} &= \frac{i}{2}\,\mathrm{Tr}\left(\overline{\psi_p}\gamma_\mu D_\mu \psi_p\right) \\ &= i\,\mathrm{Tr}\left(\overline{\chi_A}\,\tau_\mu \overset{\leftrightarrow}{D}_\mu \chi_A + \overline{\chi_B}\,\tau_\mu \overset{\leftrightarrow}{D}_\mu \chi_B + \overline{\psi}\,\tau_\mu \overset{\leftrightarrow}{D}_\mu \psi + \psi_\Phi\,\tau_\mu \overset{\leftrightarrow}{D}_\mu \overline{\psi_\Phi}\right) . \end{aligned} \tag{A.28}$$

In order to find the potential part of the fermionic $\mathcal{N}=2$ quiver gauge theory Lagrangian density we first rewrite the analogous part of the $\mathcal{N}=4$ SYM Lagrangian density (A.19). By inserting the explicit forms of the α^k, β^k matrices given in eqs. (A.7)–(A.8) into (A.19) and then decomposing the 4-component Majorana spinors into 2-component Weyl spinors according to (A.22) and finally making the substitutions (A.25), the $\mathcal{N}=4$ SYM theory result may be expressed as

$$\begin{aligned} \mathcal{L}^{\mathcal{N}=4}_{\mathrm{ferm}} = i\,\mathrm{Tr}&\left(\overline{\chi_A}\,\tau_\mu \overset{\leftrightarrow}{D}_\mu \chi_A + \overline{\chi_B}\,\tau_\mu \overset{\leftrightarrow}{D}_\mu \chi_B + \overline{\psi}\,\tau_\mu \overset{\leftrightarrow}{D}_\mu \psi + \psi_\Phi\,\tau_\mu \overset{\leftrightarrow}{D}_\mu \overline{\psi_\Phi}\right) \\ + \frac{g}{\sqrt{2}}\,\mathrm{Tr}&\Big(\overline{\chi_A}\big([A,\psi_\Phi] - [\overline{B},\overline{\psi}]\big) + \overline{\chi_B}\big([\overline{A},\overline{\psi}] + [B,\psi_\Phi]\big) \\ &\quad - \overline{\psi}\big([\overline{A},\overline{\chi_B}] - [\overline{B},\overline{\chi_A}]\big) - \psi_\Phi\big([A,\overline{\chi_A}] + [B,\overline{\chi_B}]\big) \\ &\quad + \chi_A\big([\overline{A},\overline{\psi_\Phi}] - [B,\psi]\big) + \chi_B\big([A,\psi] + [\overline{B},\overline{\psi_\Phi}]\big) \\ &\quad - \psi\big([A,\chi_B] - [B,\chi_A]\big) - \overline{\psi_\Phi}\big([\overline{A},\chi_A] + [\overline{B},\chi_B]\big) \end{aligned}$$

$$
\begin{aligned}
&+ \overline{\chi_A}\,[\overline{\Phi},\overline{\chi_B}] \;-\; \overline{\chi_B}\,[\overline{\Phi},\overline{\chi_A}] \;+\; \overline{\psi}\,[\Phi,\psi_\Phi] \;-\; \psi_\Phi\,[\Phi,\overline{\psi}] \\
&+ \chi_A\,[\Phi,\chi_B] \;-\; \chi_B\,[\Phi,\chi_A] \;+\; \psi\,[\overline{\Phi},\overline{\psi_\Phi}] \;-\; \overline{\psi_\Phi}\,[\overline{\Phi},\psi] \;\Big) . \qquad \text{(A.29)}
\end{aligned}
$$

The Weyl spinor fields $\chi_A, \chi_B, \psi_\Phi, \psi$ are the respective superpartners of A, B, Φ, A_μ. Therefore they must satisfy the $\mathbb{Z}_M$-invariance conditions

$$
\gamma^\dagger \chi_A \gamma = \omega \chi_A\,, \qquad \gamma^\dagger \chi_B \gamma = \omega^{-1} \chi_B\,, \qquad \gamma^\dagger \psi_\Phi \gamma = \psi_\Phi\,, \qquad \gamma^\dagger \psi \gamma = \psi\,. \qquad \text{(A.30)}
$$

One easily checks that these conditions are satisfied by splitting the $NM \times NM$ matrix fields of the $\mathcal{N}=4$ $\mathrm{U}(NM)$ SYM theory up into $M \times M$ block matrices whose entries are $N \times N$ matrices:

$$
\psi = \begin{pmatrix} \psi_1 & & & \\ & \psi_2 & & \\ & & \ddots & \\ & & & \psi_M \end{pmatrix}, \qquad
\chi_A = \begin{pmatrix} 0 & \chi_{A,1} & & & \\ & 0 & \chi_{A,2} & & \\ & & \ddots & \ddots & \\ & & & 0 & \chi_{A,M-1} \\ \chi_{A,M} & & & & 0 \end{pmatrix}, \qquad \text{(A.31)}
$$

$$
\chi_B = \begin{pmatrix} 0 & & & & \chi_{B,M} \\ \chi_{B,1} & 0 & & & \\ & \chi_{B,2} & \ddots & & \\ & & \ddots & 0 & \\ & & & \chi_{B,M-1} & 0 \end{pmatrix}, \qquad
\psi_\Phi = \begin{pmatrix} \psi_{\Phi,1} & & & \\ & \psi_{\Phi,2} & & \\ & & \ddots & \\ & & & \psi_{\Phi,M} \end{pmatrix}. \qquad \text{(A.32)}
$$

Here $\psi_i, \chi_{A,i}, \chi_{B,i}$ and $\psi_{\Phi,i}$ are $N \times N$ matrices (where $i = 1,\dots,M$ and we identify $i \simeq i + M$). Inserting the $\mathbb{Z}_M$-invariant forms of ψ, χ_A, χ_B and ψ_Φ given in eqs. (A.31)–(A.32) into (A.29), the spinor field part of the $\mathcal{N}=2$ quiver gauge theory Lagrangian density reads (summation over $i = 1,\dots,M$ implied)

$$
\begin{aligned}
\mathcal{L}_{\text{ferm}} &= i\,\mathrm{Tr}\Big(\overline{\chi_{A,i}}\,\tau_\mu \overset{\leftrightarrow}{D}_\mu \chi_{A,i} + \overline{\chi_{B,i}}\,\tau_\mu \overset{\leftrightarrow}{D}_\mu \chi_{B,i} + \overline{\psi_i}\,\tau_\mu \overset{\leftrightarrow}{D}_\mu \psi_i + \psi_{\Phi,i}\,\tau_\mu \overset{\leftrightarrow}{D}_\mu \overline{\psi_{\Phi,i}}\Big) \\
&+ \frac{g}{\sqrt{2}}\,\mathrm{Tr}\Big(\overline{\chi_{A,i}}\,A_{i,(i+1)}\,\psi_{\Phi,(i+1)} - \overline{\chi_{A,i}}\,\psi_{\Phi,i}\,A_{i,(i+1)} - \overline{\chi_{A,i}}\,\overline{B_{(i+1),i}}\,\overline{\psi_{i+1}} + \overline{\chi_{A,i}}\,\overline{\psi_i}\,\overline{B_{(i+1),i}} \\
&\quad + \overline{\chi_{B,i}}\,\overline{A_{i,(i+1)}}\,\overline{\psi_i} - \overline{\chi_{B,i}}\,\overline{\psi_{i+1}}\,\overline{A_{i,(i+1)}} + \overline{\chi_{B,i}}\,B_{(i+1),i}\,\psi_{\Phi,i} - \overline{\chi_{B,i}}\,\psi_{\Phi,(i+1)}\,B_{(i+1),i} \\
&\quad - \overline{\psi_{i+1}}\,\overline{A_{i,(i+1)}}\,\overline{\chi_{B,i}} + \overline{\psi_i}\,\overline{\chi_{B,i}}\,\overline{A_{i,(i+1)}} + \overline{\psi_i}\,\overline{B_{(i+1),i}}\,\overline{\chi_{A,i}} - \overline{\psi_{i+1}}\,\overline{\chi_{A,i}}\,\overline{B_{(i+1),i}} \\
&\quad - \psi_{\Phi,i}\,A_{i,(i+1)}\,\overline{\chi_{A,i}} + \psi_{\Phi,(i+1)}\,\overline{\chi_{A,i}}\,A_{i,(i+1)} - \psi_{\Phi,(i+1)}\,B_{(i+1),i}\,\overline{\chi_{B,i}} + \psi_{\Phi,i}\,\overline{\chi_{B,i}}\,B_{(i+1),i} \\
&\quad + \chi_{A,i}\,\overline{A_{i,(i+1)}}\,\overline{\psi_{\Phi,i}} - \chi_{A,i}\,\overline{\psi_{\Phi,(i+1)}}\,\overline{A_{i,(i+1)}} - \chi_{A,i}\,B_{(i+1),i}\,\psi_i + \chi_{A,i}\,\psi_{i+1}\,B_{(i+1),i} \\
&\quad + \chi_{B,i}\,A_{i,(i+1)}\,\psi_{i+1} - \chi_{B,i}\,\psi_i\,A_{i,(i+1)} + \chi_{B,i}\,\overline{B_{(i+1),i}}\,\overline{\psi_{\Phi,(i+1)}} - \chi_{B,i}\,\overline{\psi_{\Phi,i}}\,\overline{B_{(i+1),i}} \\
&\quad - \psi_i\,A_{i,(i+1)}\,\chi_{B,i} + \psi_{i+1}\,\chi_{B,i}\,A_{i,(i+1)} + \psi_{i+1}\,B_{(i+1),i}\,\chi_{A,i} - \psi_i\,\chi_{A,i}\,B_{(i+1),i} \\
&\quad - \overline{\psi_{\Phi,(i+1)}}\,\overline{A_{i,(i+1)}}\,\chi_{A,i} + \overline{\psi_{\Phi,i}}\,\chi_{A,i}\,\overline{A_{i,(i+1)}} - \overline{\psi_{\Phi,i}}\,\overline{B_{(i+1),i}}\,\chi_{B,i} + \overline{\psi_{\Phi,(i+1)}}\,\chi_{B,i}\,\overline{B_{(i+1),i}} \\
&\quad + \overline{\chi_{A,i}}\,\overline{\Phi_i}\,\overline{\chi_{B,i}} - \overline{\chi_{A,i}}\,\overline{\chi_{B,i}}\,\overline{\Phi_{i+1}} - \overline{\chi_{B,i}}\,\overline{\Phi_{i+1}}\,\overline{\chi_{A,i}} + \overline{\chi_{B,i}}\,\overline{\chi_{A,i}}\,\overline{\Phi_i} \\
&\quad + \chi_{A,i}\,\Phi_{i+1}\,\chi_{B,i} - \chi_{A,i}\,\chi_{B,i}\,\Phi_i - \chi_{B,i}\,\Phi_i\,\chi_{A,i} + \chi_{B,i}\,\chi_{A,i}\,\Phi_{i+1} \\
&\quad + \overline{\psi_i}\,\big[\Phi_i,\psi_{\Phi,i}\big] - \psi_{\Phi,i}\,\big[\Phi_i,\overline{\psi_i}\big] + \psi_i\,\big[\overline{\Phi_i},\overline{\psi_{\Phi,i}}\big] - \overline{\psi_{\Phi,i}}\,\big[\overline{\Phi_i},\psi_i\big]\Big)\,. \qquad \text{(A.33)}
\end{aligned}
$$

We conclude that the Lagrangian density of $\mathcal{N}=2$ $\mathrm{U}(N)^M$ quiver gauge theory is

$$\mathcal{L}=\mathcal{L}_{\text{scalar}}+\mathcal{L}_{\text{gauge}}+\mathcal{L}_{\text{ferm}} \tag{A.34}$$

where $\mathcal{L}_{\text{scalar}}, \mathcal{L}_{\text{gauge}}$ and $\mathcal{L}_{\text{ferm}}$ are given in eqs. (A.17), (A.18) and (A.33), respectively.

A.2 R-symmetry

The Lagrangian density of $\mathcal{N}=2$ quiver gauge theory (given in eqs. (A.34), (A.17), (A.18) and (A.33)) is invariant under global $\mathrm{SU}(2)_R\times\mathrm{U}(1)_R$ transformations. The $\mathrm{U}(1)_R$ factor of the R-symmetry group acts on the fields as

$$\begin{aligned}
A_{i,(i+1)} &\longrightarrow A_{i,(i+1)}\ , & B_{(i+1),i} &\longrightarrow B_{(i+1),i}\ , & \Phi_i &\longrightarrow e^{i\zeta}\Phi_i && \text{(A.35)}\\
\chi_{A,i} &\longrightarrow e^{-i\zeta/2}\chi_{A,i}\ , & \chi_{B,i} &\longrightarrow e^{-i\zeta/2}\chi_{B,i} &&&& \text{(A.36)}\\
\psi_i &\longrightarrow e^{i\zeta/2}\psi_i\ , & \psi_{\Phi,i} &\longrightarrow e^{-i\zeta/2}\psi_{\Phi,i}\,. &&&& \text{(A.37)}
\end{aligned}$$

The $\mathrm{U}(1)_R$ transformations of the Hermitian conjugate fields are obtained by flipping $\zeta\to-\zeta$. The Lagrangian density is manifestly invariant under the $\mathrm{U}(1)_R$ transformation.

We now move to consider the $\mathrm{SU}(2)_R$ transformations. Define the 2-component spinors

$$(\lambda_i)_a\ \equiv\ \begin{pmatrix} A_{i,(i+1)}\\ \overline{B_{(i+1),i}}\end{pmatrix}\,,\qquad\qquad (\overline{\lambda_i})^a\ \equiv\ \begin{pmatrix} \overline{A_{i,(i+1)}}\\ B_{(i+1),i}\end{pmatrix}\,. \tag{A.38}$$

Under $\sigma\in\mathrm{SU}(2)_R$ these spinors have the transformations

$$\begin{aligned}
(\lambda_i)_a &\longrightarrow \sigma_a{}^b\,(\lambda_i)_b && \text{(A.39)}\\
(\overline{\lambda_i})^a &\longrightarrow (\overline{\lambda_i})^b\,\overline{\sigma}_b{}^a\,. && \text{(A.40)}
\end{aligned}$$

Note that $(\overline{\lambda_i})_a=\epsilon_{ab}(\overline{\lambda_i})^b$ has the transformation

$$(\overline{\lambda_i})_a\ \longrightarrow\ \epsilon_{ab}\,\overline{\sigma}_c{}^b\,\epsilon^{dc}\,(\overline{\lambda_i})_d=\sigma_a{}^d(\overline{\lambda_i})_d \tag{A.41}$$

where the equality follows by using $\sigma\in\mathrm{SU}(2)_R$. Thus, $(\lambda_i)_a$ and $(\overline{\lambda_i})_a$ are $\mathrm{SU}(2)_R$ doublets. To exhibit the $\mathrm{SU}(2)_R$ invariance of the Lagrangian density we define $\mathrm{SU}(2)_R$ invariants such as

$$(\lambda_i)_a\,(\overline{\lambda_i})^a=-\epsilon^{ab}(\lambda_i)_a(\overline{\lambda_i})_b=-A_{i,(i+1)}\overline{A_{i,(i+1)}}-\overline{B_{(i+1),i}}B_{(i+1),i} \tag{A.42}$$

and write the Lagrangian density in terms of these. For $\mathcal{N}=2$ quiver gauge theory the bi-fundamental scalars and the adjoint fermions are organized into $\mathrm{SU}(2)_R$ doublets as follows

$$(\lambda_i)_a\ \equiv\ \begin{pmatrix} A_{i,(i+1)}\\ \overline{B_{(i+1),i}}\end{pmatrix}\,,\qquad\qquad (\overline{\lambda_i})_a\ \equiv\ \begin{pmatrix} -B_{(i+1),i}\\ \overline{A_{i,(i+1)}}\end{pmatrix} \tag{A.43}$$

$$(\chi_i)_a\ \equiv\ \begin{pmatrix} \overline{\psi_i}\\ \psi_{\Phi,i}\end{pmatrix}\,,\qquad\qquad (\overline{\chi_i})_a\ \equiv\ \begin{pmatrix} -\overline{\psi_{\Phi,i}}\\ \psi_i\end{pmatrix}\,. \tag{A.44}$$

The scalar field Lagrangian density written in terms of the $\mathrm{SU}(2)_R$ doublets takes the following form[27]

$$
\begin{aligned}
\mathcal{L}_{\text{scalar}} = \sum_{i=1}^{M} \Big[& \operatorname{Tr}\left(\epsilon^{ab}(D_\mu\lambda_i)_a(D_\mu\overline{\lambda_i})_b + D_\mu\Phi_i D_\mu\overline{\Phi_i}\right) \\
& + \frac{1}{2}g^2 \operatorname{Tr}\left(\epsilon^{ab}(\lambda_i)_a(\overline{\lambda_i})_b - \epsilon^{ab}(\overline{\lambda_{i-1}})_a(\lambda_{i-1})_b + [\Phi_i, \overline{\Phi_i}]\right)^2 \\
& - 2g^2 \operatorname{Tr}\left(\epsilon^{ab}(\lambda_i)_a(\overline{\lambda_i})_b - \epsilon^{ab}(\overline{\lambda_{i-1}})_a(\lambda_{i-1})_b\right)^2 \\
& - 2g^2 \operatorname{Tr}\left(\epsilon^{ab}(\lambda_i)_a(\overline{\lambda_i})_b\epsilon^{cd}(\overline{\lambda_{i-1}})_c(\lambda_{i-1})_d + \epsilon^{ab}(\lambda_{i-1})_a(\lambda_i)_b\epsilon^{cd}(\overline{\lambda_i})_c(\overline{\lambda_{i-1}})_d\right) \\
& + 2g^2 \operatorname{Tr}\left(\epsilon^{ab}(\overline{\lambda_i})_a(\lambda_i)_b\epsilon^{cd}(\overline{\lambda_i})_c(\lambda_i)_d + \epsilon^{ab}(\lambda_i)_a(\overline{\lambda_i})_b\epsilon^{cd}(\lambda_i)_c(\overline{\lambda_i})_d\right) \\
& - 2g^2 \operatorname{Tr}\left(\epsilon^{ab}(\lambda_i)_a\Phi_{i+1}(\overline{\lambda_i})_b\overline{\Phi_i} + \epsilon^{ab}(\lambda_i)_a\overline{\Phi_{i+1}}(\overline{\lambda_i})_b\Phi_i\right) \\
& + 2g^2 \operatorname{Tr}\left(\epsilon^{ab}(\lambda_i)_a\overline{\Phi_{i+1}}\Phi_{i+1}(\overline{\lambda_i})_b + \epsilon^{ab}(\lambda_i)_a(\overline{\lambda_i})_b\overline{\Phi_i}\Phi_i\right)\Big] \,.
\end{aligned}
\tag{A.45}
$$

The spinor field Lagrangian density written in terms of the $\mathrm{SU}(2)_R$ doublets takes the following form

$$
\begin{aligned}
\mathcal{L}_{\text{ferm}} = \sum_{i=1}^{M} \Big[& i \operatorname{Tr}\left(\overline{\chi_{A,i}}\,\tau_\mu \overleftrightarrow{D}_\mu \chi_{A,i} + \overline{\chi_{B,i}}\,\tau_\mu \overleftrightarrow{D}_\mu \chi_{B,i} + \epsilon^{cd}(\chi_i)_c(\tau_\mu \overleftrightarrow{D}_\mu \overline{\chi_i})_d\right) \\
& + \frac{g}{\sqrt{2}} \operatorname{Tr}\Big(\epsilon^{cd}\big\{\chi_{A,i}(\overline{\lambda_i})_c,\, (\overline{\chi_i})_d\big\} + \epsilon^{cd}\big\{\chi_{A,i},\, (\overline{\chi_{i+1}})_c(\overline{\lambda_i})_d\big\} \\
& \qquad + \epsilon^{cd}\big\{\overline{\chi_{A,i}}\,(\lambda_i)_c,\, (\chi_{i+1})_d\big\} + \epsilon^{cd}\big\{\overline{\chi_{A,i}},\, (\chi_i)_c(\lambda_i)_d\big\} \\
& \qquad + \epsilon^{cd}\big\{\chi_{B,i}(\lambda_i)_c,\, (\overline{\chi_{i+1}})_d\big\} + \epsilon^{cd}\big\{\chi_{B,i},\, (\overline{\chi_i})_c(\lambda_i)_d\big\} \\
& \qquad - \epsilon^{cd}\big\{\overline{\chi_{B,i}}\,(\overline{\lambda_i})_c,\, (\chi_i)_d\big\} - \epsilon^{cd}\big\{\overline{\chi_{B,i}},\, (\chi_{i+1})_c(\overline{\lambda_i})_d\big\} \\
& \qquad + \epsilon^{cd}\big\{(\chi_i)_c\Phi_i,\, (\chi_i)_d\big\} + \epsilon^{cd}\big\{(\overline{\chi_i})_c\overline{\Phi_i},\, (\overline{\chi_i})_d\big\} \\
& \qquad + \big\{\chi_{A,i}\Phi_{i+1},\, \chi_{B,i}\big\} + \big\{\overline{\chi_{A,i}}\,\overline{\Phi_i},\, \overline{\chi_{B,i}}\big\} \\
& \qquad - \big\{\chi_{B,i}\Phi_i,\, \chi_{A,i}\big\} - \big\{\overline{\chi_{B,i}}\,\overline{\Phi_{i+1}},\, \overline{\chi_{A,i}}\big\}\Big)\Big] \,.
\end{aligned}
\tag{A.46}
$$

These results are conveniently summarized in table 2 which lists the R-charges of all the fields in $\mathcal{N} = 2$ quiver gauge theory.

Here the generators of $\mathfrak{su}(2)_R$ are taken in the fundamental representation and chosen as $\frac{1}{2}(\sigma_x, \sigma_y, \sigma_z)$. The R-charges of the corresponding Hermitian conjugate fields are obtained by simply changing the signs of the $\mathrm{U}(1)_R$ and $\mathrm{SU}(2)_R$ charges.

B. Bosonic and fermionic fluctuation determinants

In this appendix we present some technical details of the computation of the 1-loop quantum effective action given in section 5. More specifically, we explain here how to evaluate the fluctuation determinants arising from path integrating over the fluctuating fields.

[27] Note that the term $R^{-2} \operatorname{Tr}\left(\epsilon^{ab}(\lambda_i)_a(\overline{\lambda_i})_b + \Phi_i\overline{\Phi_i}\right)$ describing the conformal coupling of the scalar fields to the curvature has been omitted here.

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	$A_{i,(i+1)}$	$B_{(i+1),i}$	Φ_i	$A_{\mu i}$	$\chi_{A,i}$	$\chi_{B,i}$	$\psi_{\Phi,i}$	ψ_i
$\mathrm{U}(1)_R$	0	0	1	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
$\mathrm{SU}(2)_R$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$

Table 2: R-charges for the bosonic and fermionic fields

B.1 Bosonic case

The fluctuation operators $\Box_g^{mn}, \Box_{\mathbf{A}}^{mn}, \Box_{\mathbf{B}}^{mn}$ and $\Box_{\mathbf{\Phi}}^{mn}$ in eqs. (5.21)–(5.22) are given as below.

$$(\Box_g^{mn})_{ij} = \begin{cases} -2\Big((\overline{a_{(i-1),i}})_{nn}\,(a_{(i-1),i})_{mm} + (b_{i,(i-1)})_{nn}\,(\overline{b_{i,(i-1)}})_{mm}\Big) & \text{for } j=i-1 \\ -\partial^2 - 2i(\alpha_i^n - \alpha_i^m)\,\partial_0 + (\alpha_i^n - \alpha_i^m)^2 & \\ \quad +2\Big((a_{i,(i+1)})_{nn}\,(\overline{a_{i,(i+1)}})_{nn} + (a_{(i-1),i})_{mm}\,(\overline{a_{(i-1),i}})_{mm} & \\ \qquad + (b_{i,(i-1)})_{nn}\,(\overline{b_{i,(i-1)}})_{nn} + (b_{(i+1),i})_{mm}\,(\overline{b_{(i+1),i}})_{mm} & \\ \qquad + \big((\phi_i)_{nn} - (\phi_i)_{mm}\big)\big((\overline{\phi_i})_{nn} - (\overline{\phi_i})_{mm}\big)\Big) & \text{for } j=i \\ -2\Big((a_{i,(i+1)})_{nn}\,(\overline{a_{i,(i+1)}})_{mm} + (\overline{b_{(i+1),i}})_{nn}\,(b_{(i+1),i})_{mm}\Big) & \text{for } j=i+1 \end{cases} \tag{B.1}$$

and

$$(\Box_{\mathbf{A}}^{mn})_{ij} = \begin{cases} -2\Big((\overline{a_{i,(i+1)}})_{nn}\,(a_{(i-1),i})_{mm} + (b_{(i+1),i})_{nn}\,(\overline{b_{i,(i-1)}})_{mm}\Big) & \text{for } j=i-1 \\ -\partial^2 - 2i(\alpha_{i+1}^n - \alpha_i^m)\,\partial_0 + (\alpha_{i+1}^n - \alpha_i^m)^2 + R^{-2} & \\ \quad +\ 2\Big((a_{i,(i+1)})_{nn}\,(\overline{a_{i,(i+1)}})_{nn} + (a_{i,(i+1)})_{mm}\,(\overline{a_{i,(i+1)}})_{mm} & \\ \qquad + (b_{(i+1),i})_{nn}\,(\overline{b_{(i+1),i}})_{nn} + (b_{(i+1),i})_{mm}\,(\overline{b_{(i+1),i}})_{mm} & \\ \qquad + \big((\phi_{i+1})_{nn} - (\phi_i)_{mm}\big)\big((\overline{\phi_{i+1}})_{nn} - (\overline{\phi_i})_{mm}\big)\Big) & \text{for } j=i \\ -2\Big((a_{(i+1),(i+2)})_{nn}\,(\overline{a_{i,(i+1)}})_{mm} + (\overline{b_{(i+2),(i+1)}})_{nn}\,(b_{(i+1),i})_{mm}\Big) & \text{for } j=i+1 \end{cases} \tag{B.2}$$

and

$$(\Box_{\mathbf{B}}^{mn})_{ij} = \begin{cases} -2\Big((\overline{a_{(i-1),i}})_{nn}\,(a_{i,(i+1)})_{mm} + (b_{i,(i-1)})_{nn}\,(\overline{b_{(i+1),i}})_{mm}\Big) & \text{for } j=i-1 \\ -\partial^2 - 2i(\alpha_i^n - \alpha_{i+1}^m)\,\partial_0 + (\alpha_i^n - \alpha_{i+1}^m)^2 + R^{-2} & \\ \quad +\ 2\Big((a_{i,(i+1)})_{nn}\,(\overline{a_{i,(i+1)}})_{nn} + (a_{i,(i+1)})_{mm}\,(\overline{a_{i,(i+1)}})_{mm} & \\ \qquad + (b_{(i+1),i})_{nn}\,(\overline{b_{(i+1),i}})_{nn} + (b_{(i+1),i})_{mm}\,(\overline{b_{(i+1),i}})_{mm} & \\ \qquad + \big((\phi_i)_{nn} - (\phi_{i+1})_{mm}\big)\big((\overline{\phi_i})_{nn} - (\overline{\phi_{i+1}})_{mm}\big)\Big) & \text{for } j=i \\ -2\Big((a_{i,(i+1)})_{nn}\,(\overline{a_{(i+1),(i+2)}})_{mm} + (\overline{b_{(i+1),i}})_{nn}\,(b_{(i+2),(i+1)})_{mm}\Big) & \text{for } j=i+1 \end{cases} \tag{B.3}$$

and

$$
(\Box^{mn}_{\mathbf{\Phi}})_{ij} = \begin{cases} -2\Big((\overline{a_{(i-1),i}})_{nn}\,(a_{(i-1),i})_{mm} + (b_{i,(i-1)})_{nn}\,(\overline{b_{i,(i-1)}})_{mm}\Big) & \text{for } j = i-1 \\[2ex] -\partial^2 - 2i(\alpha^n_i - \alpha^m_i)\,\partial_0 + (\alpha^n_i - \alpha^m_i)^2 + R^{-2} \\ \quad + 2\Big((a_{i,(i+1)})_{nn}\,(\overline{a_{i,(i+1)}})_{nn} + (a_{(i-1),i})_{mm}\,(\overline{a_{(i-1),i}})_{mm} \\ \qquad + (b_{i,(i-1)})_{nn}\,(\overline{b_{i,(i-1)}})_{nn} + (b_{(i+1),i})_{mm}\,(\overline{b_{(i+1),i}})_{mm} \\ \qquad + \big((\phi_i)_{nn} - (\phi_i)_{mm}\big)\,\big((\overline{\phi_i})_{nn} - (\overline{\phi_i})_{mm}\big)\Big) & \text{for } j = i \\[2ex] -2\Big((a_{i,(i+1)})_{nn}\,(\overline{a_{i,(i+1)}})_{mm} + (\overline{b_{(i+1),i}})_{nn}\,(b_{(i+1),i})_{mm}\Big) & \text{for } j = i+1 \end{cases} \tag{B.4}
$$

In the general vacuum (5.10)–(5.13) these operators are tridiagonal, periodically continued matrices (assuming $M \geq 3$). The determinant of this class of matrices was considered in ref. [59] (appendix B) who found the following result, valid for $M \geq 3$:

$$
\begin{aligned}
\det \Box^{mn} = \operatorname{tr} \prod_{i=M}^{1} &\begin{pmatrix} (\Box^{mn})_{ii} & -(\Box^{mn})_{i,(i-1)}(\Box^{mn})_{(i-1),i} \\ 1 & 0 \end{pmatrix} \\
&+ (-1)^{M+1} \operatorname{tr} \prod_{i=M}^{1} \begin{pmatrix} (\Box^{mn})_{i,(i-1)} & 0 \\ 0 & (\Box^{mn})_{(i-1),i} \end{pmatrix} .
\end{aligned} \tag{B.5}
$$

The inverse order of the initial and final indices on the product symbol indicates that the matrix with the highest index i is on the left of the product.

Fortunately, in the vacuum (5.14)–(5.17) the fluctuation determinants take a much simpler form. Namely, using (5.14)–(5.17), the operators $\Box^{mn}_g, \Box^{mn}_{\mathbf{A}}, \Box^{mn}_{\mathbf{B}}, \Box^{mn}_{\mathbf{\Phi}}$ (for fixed m, n) can be written in the particular form below, and there is a simple closed expression for the determinant.[28] That is, defining $\omega \equiv e^{2\pi i/M}$, we have the determinant formula

$$
\det \begin{pmatrix} \xi & -\eta & & & -\omega^{-k(M-1)}\overline{\eta} \\ -\overline{\eta} & \xi & -\omega^k \eta & & \\ & -\omega^{-k}\overline{\eta} & \xi & \ddots & \\ & & \ddots & \ddots & -\omega^{k(M-2)}\eta \\ -\omega^{k(M-1)}\eta & & -\omega^{-k(M-2)}\overline{\eta} & & \xi \end{pmatrix} = \prod_{i=1}^{M} \left(\xi - \omega^i \eta - \omega^{-i}\overline{\eta}\right) . \tag{B.6}
$$

Note in particular that the phases ω^k on the left hand side cancel out. Therefore, for any value of $k \in \mathbb{Z}$ in (5.14)–(5.15), one obtains the same result for the fluctuation determinants.

B.2 Fermionic case

In order to compute the fluctuation determinant arising from path integrating over the

[28] To prove the formula, note first that the powers of ω^k appearing in the super- and subdiagonal mutually cancel according to (B.5), so the determinant is independent of the value of k. Putting $k = 0$, the formula (B.6) is a special case of eq. (A.1) in ref. [20].

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fermionic fluctuations we must first introduce some notation:

$$
J_k^+(\omega) \equiv \begin{pmatrix} 0 & 1 & & & \\ & 0 & \omega^k & & \\ & & 0 & \ddots & \\ & & & \ddots & \omega^{k(M-2)} \\ \omega^{k(M-1)} & & & & 0 \end{pmatrix}, \quad J_k^-(\omega) \equiv \begin{pmatrix} 0 & & & & \omega^{k(M-1)} \\ 1 & 0 & & & \\ & \omega^k & 0 & & \\ & & \ddots & \ddots & \\ & & & \omega^{k(M-2)} & 0 \end{pmatrix} \tag{B.7}
$$

and

$$
w_1 \equiv a = \langle A \rangle \,, \qquad \Omega_1 \equiv J_k^+(\omega) \tag{B.8}
$$
$$
w_2 \equiv b = \langle B \rangle \,, \qquad \Omega_2 \equiv J_k^-(\omega^{-1}) \tag{B.9}
$$
$$
w_3 \equiv \phi = \langle \Phi \rangle \,, \qquad \Omega_3 \equiv \mathbf{1}_M \tag{B.10}
$$

where it is implied that A, B, Φ take the $\mathbb{Z}_M$ projection invariant forms given in (A.15)–(A.16).

The fluctuation operator $\mathbf{D}_{ij}^{mn}$ in (5.26) is as given below (where $c, d = 1, 2, 3$)

$$
\mathbf{D}_{ij}^{mn} \equiv \begin{pmatrix} \frac{i}{2}\delta_{pq}\overline{\tau_\mu}\big(\partial_\mu + \delta_{\mu 0}(\alpha_i^n - \alpha_i^m)\big)\delta_{ij} & \begin{matrix} -\frac{1}{\sqrt{2}}\alpha_{pq}^c \left[\big((w_c)_{nn} + (\overline{w_c})_{nn}\big) - \big((w_c)_{mm}\Omega_c + (\overline{w_c})_{mm}\Omega_c^{-1}\big)\right]_{ij} \\ +\frac{1}{\sqrt{2}}\beta_{pq}^c \left[\big((w_c)_{nn} - (\overline{w_c})_{nn}\big) - \big((w_c)_{mm}\Omega_c - (\overline{w_c})_{mm}\Omega_c^{-1}\big)\right]_{ij} \end{matrix} \\ \begin{matrix} -\frac{1}{\sqrt{2}}\alpha_{pq}^d \left[\big((w_d)_{nn} + (\overline{w_d})_{nn}\big) - \big((w_d)_{mm}\Omega_d + (\overline{w_d})_{mm}\Omega_d^{-1}\big)\right]_{ij} \\ -\frac{1}{\sqrt{2}}\beta_{pq}^d \left[\big((w_d)_{nn} - (\overline{w_d})_{nn}\big) - \big((w_d)_{mm}\Omega_d - (\overline{w_d})_{mm}\Omega_d^{-1}\big)\right]_{ij} \end{matrix} & \frac{i}{2}\delta_{pq}\tau_\nu\big(\partial_\nu + \delta_{\nu 0}(\alpha_i^n - \alpha_i^m)\big)\delta_{ij} \end{pmatrix} \tag{B.11}
$$

The reason why the w_c entries labelled by the gauge index m have additional factors of Ω_c compared to the entries labelled by n comes from the commutator structure of the Yukawa coupling (see (A.29)). Namely, when taking the trace over the gauge indices, the w_c entries labelled by n correspond to the terms where a scalar field appears between two spinor fields, whereas those labelled with m correspond to the terms where the scalar field appears to the right of both spinor fields. After substituting the orbifold projection invariant forms given in eqs. (A.15)–(A.16) and (A.31)–(A.32), the bifundamental scalar VEV's will couple different pairs of spinor fields depending on whether the VEV appears between the spinor fields or to the right of them in the Yukawa coupling. Since the scalar VEV's are mutually related through the vacuum (5.14)–(5.17), this can be compensated for by appropriately multiplying factors of Ω_c.

To compute the result of the path integrations it is convenient to define (for a fixed c)

$$
F_c \equiv \big((w_c)_{nn} + (\overline{w_c})_{nn}\big) - \big((w_c)_{mm}\Omega_c + (\overline{w_c})_{mm}\Omega_c^{-1}\big) \tag{B.12}
$$
$$
G_c \equiv \big((w_c)_{nn} - (\overline{w_c})_{nn}\big) - \big((w_c)_{mm}\Omega_c - (\overline{w_c})_{mm}\Omega_c^{-1}\big) \,. \tag{B.13}
$$

Noting that

$$
\big[F_c, F_d\big] = 0 \,, \qquad \big[F_c, G_d\big] = 0 \,, \qquad \big[G_c, G_d\big] = 0 \tag{B.14}
$$

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one finds, by using the (anti)commutation relations (A.6) for α^c and β^d, that the result of the path integrations over the fermionic fluctuations $(\lambda_p)_i, (\overline{\lambda_p})_i$ is

$$
\begin{aligned}
\det(\mathbf{D}^{mn}_{ij}) &= \det\Bigg(-\left(\partial_\mu + i\delta_{\mu 0}(\alpha^n_i - \alpha^m_i)\right)^2 \\
&\qquad - \frac{1}{2}\left(\frac{1}{2}\{\alpha^c,\alpha^d\}_{pr}F_cF_d + [\alpha^c,\beta^d]_{pr}F_cG_d - \frac{1}{2}\{\beta^c,\beta^d\}_{pr}G_cG_d\right)\Bigg) && \text{(B.15)} \\
&= \det\left(\left(i\partial_\mu - \delta_{\mu 0}(\alpha^n_i - \alpha^m_i)\right)^2 + \frac{1}{2}(F_cF_d - G_cG_d)\,\delta_{cd}\delta_{pr}\right) && \text{(B.16)} \\
&= \det \Delta_{ij}\,. && \text{(B.17)}
\end{aligned}
$$

Here we have defined the $M \times M$ matrix (labelled by $i, j = 1, \ldots, M$)

$$
\Delta_{ij} = \begin{cases}
-2\left((a_{1,2})_{nn}\,(\overline{a_{1,2}})_{mm} + (\overline{b_{2,1}})_{nn}\,(b_{2,1})_{mm}\right)\omega^{-(i-2)k} & \text{for } j = i-1 \\[2ex]
-\partial^2 - 2i(\alpha^n_1 - \alpha^m_1)\partial_0 + (\alpha^n_1 - \alpha^m_1)^2 & \\
\quad + 2\Big((a_{1,2})_{nn}\,(\overline{a_{1,2}})_{nn} + (a_{1,2})_{mm}\,(\overline{a_{1,2}})_{mm} & \\
\qquad + (b_{2,1})_{nn}\,(\overline{b_{2,1}})_{nn} + (b_{2,1})_{mm}\,(\overline{b_{2,1}})_{mm} & \\
\qquad + \left((\phi_1)_{nn} - (\phi_1)_{mm}\right)\left((\overline{\phi_1})_{nn} - (\overline{\phi_1})_{mm}\right)\Big) & \text{for } j = i \\[2ex]
-2\left((\overline{a_{1,2}})_{nn}\,(a_{1,2})_{mm} + (b_{2,1})_{nn}\,(\overline{b_{2,1}})_{mm}\right)\omega^{(i-1)k} & \text{for } j = i+1
\end{cases}
\qquad \text{(B.18)}
$$

where we have used (5.14)–(5.17) to arrive at the equality (B.17). Applying the determinant formula (B.6) and using (5.14)–(5.17) again one finds, after taking the traces over the fermionic Matsubara frequencies $\omega_k \equiv \frac{(2k+1)\pi}{\beta}$ and over the S^3 spherical harmonics, the expression (5.27).

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